Search

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The Set Up of Discrete Time Model

- The decision maker samples sequentially from the distribution F. I.e., he faces a sequence $\{x_i\}$ of i.i.d. random variables with cumulative distribution function F
- Each observation costs c (a fixed positive number).
- He can stop search at any time $n \in \{1, 2, 3, ...\}$.
- Let y_n be the prize received if search is stopped after n samples.
  For search with recall, $y_n = \max\{x_1, x_2, ..., x_n\}$.
  For search without recall, $y_n = x_n$.
- Payoff for the decision maker stopping after n samples is denoted Y_n where

$$Y_n = y_n - nc$$

  If the decision maker discount the the sampling time with discount factor $\delta < 1$, then the payoff is

$$Y_n = \delta^{n-1} y_n - \sum_{i=1}^{n} \delta^{i-1} c = \delta^{n-1} y_n - \frac{1-\delta^n}{1-\delta} \cdot c.$$

The Problem

- A **stopping rule** prescribes a rule after what sequences of observations to stop the sampling.
- A **stopping time** resulting from a stopping rule is the integer n after which the sampling will stop, if that stopping rule is invoked. *The stopping time is a random variable* (because samples are random variables).
- A stopping rule, S, determines a random stopping time $N(S)$.
- The problem the decision maker is facing is to find and use a stopping rule, S, that maximizes its expected payoff,

$$E[Y_{N(S)}].$$

- The solution to this problem (a stopping rule), is called an **optimal stopping rule**.

An Optimal Stopping Rule is a Reservation Value Rule

- A **reservation value rule** is a stopping rule which prescribes to stop sampling after n observations if and only if $y_n \geq y$. The level y is called the reservation value.
- It turns out that under certain conditions, a optimal stopping rule is a reservation value rule.

Proposition

If $E[x_i^2]$ is finite, then there exists an optimal stopping rule. It is a reservation value rule.

See Maurice DeGroot (1970, chapter 13.9) for the proof.

Some Properties of the Reservation Value Rule (I)

- Let $N(y)$ be the stopping time associated with the reservation value rule with reservation value y. Since probability of $N(y)$ being i is $F(y)^{i-1}(1-F(y))$, we can calculate the expected value of $N(y)$ as follows;

$$E[N(y)] = \sum_{i=1}^{\infty} iF(y)^{i-1}(1-F(y)) = 1/(1-F(y)). \tag{1}$$

- Let $V(y)$ be the expected value of reservation value rule with reservation value y. I.e., $V(y) \equiv E[Y_{N(y)}]$. This can be calculated as follows;

$$\begin{aligned} V(y) = E[Y_{N(y)}] &= E[y_n - N(y)c] = E[x|x \geq y] - cE[N(y)] \\ &= \int_y^{\infty} x \frac{dF(x)}{1-F(y)} - \frac{c}{1-F(y)}. \end{aligned} \tag{2}$$

Some Properties of the Reservation Value Rule (II)

Proposition

If F has a connected support, then the optimal policy has a reservation value y^* such that $y^* = V(y^*)$.

- The optimal reservation value y^* solves the equation,

$$c = \int_{y^*}^{\infty} (x - y^*) dF(x) \tag{3}$$

- Comparative statics results;
  (1) If the search cost c is higher, then the reservation value y^* becomes lower.
  (2) If cumulative distribution function for samples G is a mean preserving spread of F, the reservation value under G is higher than that under F. (G is mean preserving spread of F if (i) they have the same mean and (ii) $\int_{-\infty}^{y} G(x) - F(x) dx \geq 0$ for all y)

Homework I

Homework I

- Prove comparative statics results in the previous slide.
- Derive expression for $V(y)$ (equivalent of expression (2)) if decision maker discounts time with discount factor $\delta < 1$.
- Using above result, perform a comparative static of change in δ on the optimal reservation value.
- Consider the case where time ends at period T. Prove that optimal strategy is again a reservation value rule. (Hint: solve the problem from the last period and using this result solve for the problem before the last period, and so on... to the beginning.)

The Set Up of Continuous Time Model

- Time is continuous and is denoted by $t \in [0, \infty)$
- The decision maker seek observations x_i from the distribution F arriving according to a Poisson arrival process with parameter a. Thus, the probability of arrival in a short interval of time dt is given by $a \cdot dt$.
- In order to draw observations, the decision maker has to pay search costs $c \cdot dt$ for each short time interval dt.
- Benefits are discounted at a constant discount rate r. Thus, the present value of a dollar after the passage of time interval T is $\exp(-rT)$.
- Let y_T be the prize received if search is stopped after time interval T.
- Payoff for the decision maker if search is stopped after time interval T is given by,

$$\exp(-rT)y_T - \int_0^T \exp(-r\tau)c \cdot d\tau.$$

The Expected Value under Reservation Value Rule (I)

- Given a stopping rule S, stopping time under stopping rule S is denoted by $T(S)$. This is a random variable.
- We denote $T(y)$ stopping time under reservation value rule with reservation value y.
- Expected payoff for the decision maker under reservation value rule with reservation value y which we denote $v(y)$ is given by,

$$v(y) \equiv E[\exp(-rT(y)) \cdot y_{T(y)} - \int_0^{T(y)} \exp(-r\tau)c \cdot d\tau].$$

- It turns out that optimal reservation value is y^* such that

$$y^* = v(y^*).$$

The Expected Value under Reservation Value Rule (II)

- We divide future into a short time interval Δt and the time after that. Possible events during this short time interval are; (i) only one observation is made the sample is above the reservation value y (with probability $a \cdot \Delta t \cdot (1 - F(y))$), (ii) only one observation is made but the sample is below the reservation value y (with probability $a \cdot \Delta t F(y)$), (iii) no observation is made (with probability $(1 - a \cdot \Delta t - o(\Delta t))$), or (iv) others (with probability $o(\Delta t)$), where $o(\Delta t)$ stands for polynomials with order higher than or equal to Δt^2. This implies that $\lim_{\Delta t \to 0} o(\Delta t)/\Delta t = 0$.)
- Expected payoff for the decision maker under reservation value rule with reservation value y, can be calculated as,

$$
\begin{aligned}
v(y) \quad = \quad & E[x|x \geq y][1 - F(y)]a \cdot \Delta t - c \cdot \Delta t \\
& + [1 - a \cdot \Delta t - o(\Delta t) + aF(y)\Delta t] \exp(-r\Delta t) \cdot v(y) \\
& + o(\Delta t).
\end{aligned}
$$

The Expected Value under Reservation Value Rule (III)

$$\begin{aligned}
v(y) &= [1-F(y)]a\Delta t \int_y^\infty x \frac{dF(x)}{1-F(y)} - c\Delta t \\
&\quad +(1 - a\Delta t + aF(y)\Delta t)(1 - r\Delta t)v(y) + o(\Delta t) \\
&= a\Delta t \int_y^\infty x dF(x) - c\Delta t \\
&\quad +(1 - a\Delta t + aF(y)\Delta t - r\Delta t)v(y) + o(\Delta t) \\
(r &+ a \cdot (1 - F(y)))\Delta t \cdot v(y) = a\Delta t \int_y^\infty x dF(x) - c\Delta t + o(\Delta t) \\
v(y) &= \frac{a \int_y^\infty x dF(x) - c}{r + a \cdot (1 - F(y))} + o(\Delta t)/\Delta t
\end{aligned}$$

Taking the limit $\Delta t \to 0$, we obtain

$$v(y) = \frac{a \int_y^\infty x dF(x) - c}{r + a \cdot (1 - F(y))} \tag{4}$$

The Optimal Reservation Value

- The optimal reservation value y^* is a reservation value such that
$$y^* = v(y^*).$$
This can be shown by maximizing (4) with respect to y. The first order condition for this is as follows;
$$\begin{aligned} v'(y^*) &= 0, \\ \frac{-ay^* dF(y^*)}{r + a \cdot (1 - F(y^*))} + \frac{[a \int_{y^*}^{\infty} x dF(x) - c] a dF(y^*)}{[r + a \cdot (1 - F(y^*))]^2} &= 0, \\ -y^* + v(y^*) &= 0. \end{aligned}$$

- It is straightforward to rewrite the value function (4) under the optimal reservation value y^* as follows:
$$rv(y^*) = ry^* = a \int_{y^*}^{\infty} [x - y^*] dF(x) - c. \tag{5}$$

Diamond (1970)'s Model: The Set Up

- Players:
  - There are large number, n, of firms producing a homogenous product at no cost.
  - There are large number of identical consumers. Each wants one unit of goods and is willing to pay up to v.
- The sequence of the game:
  - Firms choose their prices simultaneously. Firm $i \in \{1, 2, ..., n\}$ sets P_i.
  - Consumers learn about the distribution of prices.
  - At the beginning, each consumer gets, at no cost, a price quote from one firm chosen at random. I.e., $1/n$ proportion of all consumers learn about P_1 the price set by firm 1, another $1/n$ proportion of all consumers learn about P_2 and so on.
  - Next, each consumer may get additional quotes sequentially at the cost c per quote. Each of these quotes is a random draw from the set of all firms.
  - Finally, each consumer decides whether and from which firm to buy.

The Subgame Perfect Equilibrium of the Game

- Players' Strategies:
  - Firm i's strategy is a price P_i
  - A consumer's strategy prescribes a stopping rule as a function of the price distribution and the free quote he received.
- A **Subgame Perfect Equilibrium (SPE)** of this game is a choice of strategies for firms and consumers such that
  - each firm sets price in order to maximize its profit given the strategies of all others;
  - and for any price distribution and an initial quote, each consumer's strategy prescribes the optimal search rule.

Proposition:
If n is sufficiently large, $n > 1 + v/c$, then the unique SPE is $p_i = v$ for all i.

Proof of Proposition

- **Each firm can guarantee to itself some positive profit** by choosing a positive $p < c$.

  Because it is better for any consumer to accept that price quote than to continue searching for another.
- **No consumer searches.**

  Look at firm with the maximal price. By above, this firm must be making a positive profit. Therefore, either all of consumers who sample it buy, or some fraction of those who sample it continue to search. The latter case means that the maximal price is exactly the reservation value. Thus, this firm can retain all its customers and increase profits by cutting its price very slightly. But this may not be true in equilibrium, so that all those who sample the maximal price buy. Since it is true for maximal price it is true for any other price.
- **Firms have an incentive to raise price if it is set below v.**

  Suppose that some price is below v. Let p_i be the minmal price. Since there is no search, this firm will profit from changing the price to $p = p_i + \epsilon$ where $\epsilon < c$. This is because none of its customers will leave this firm. And this firm is not getting other customers anyway.
- **All prices are equal to v in an equilibrium.**

  Suppose all firms are setting price equal to v. Let us check that there is no incentive for any firm to lower the price. The only reason a firm may want to lower the price that it could induce search. However, by the choice of n, even if such firm lowers its price to zero, no consumer will search since the expected search cost, nc exceeds v.

Comments

- The result is not an artifact of this special example. It caputures an important element that appears in many search models.
- Even small search costs can lead to the monopoly outcome in a seemingly competitive environment.
- The imperfect information itself cannot explain price dispersion in a symmetric situation with sequential search. To explain price dispersion, there have to be asymmetries or non-sequential search.

Burdett and Judds's (1983) Model: The Set Up

- Players: Continuum of N firms and M consumers. There are $m \equiv M/N$ consumers per firm.
- Consumers are identical and each wants one unit. Each has reservation value v, decides on sample of n firms at the cost $c \cdot (n-1)$, and buys at the lowest sampled price if it is lower than v (and chooses with equal probability if a few firms quote the lowest price).
- Firms choose price p. Each produces any quantity at 0 cost. The profit of a firm charging p is denoted by $H(p)$.
- Suppose consumers' behavior is fixed. Let q_n denote the fraction of the consumer population who make n observations. And suppose that $\sum_{n=1}^{\infty} q_n \cdot n \infty$.

Firms' Behavior (I)

Let us consider firms' behavior.

Definition: A **firm equilibrium** is a pair $(F(\cdot), H)$ where F is the price distribution and H a number such that
(a) $H(p) = H$ for all p in the support of F;
(b) $H(p) \leq H$, for all p.

Lemma: If $q_1 < 1$ and (F, H) a firm equilibrium, F is either continuous with connected support or concentrated at 0. Also $F(v) = 1$.

Proof of Lemma:
Step 1: F cannot have a mass point at p' in $(0, v]$ since then a firm charging p' can profit by slightly lowering the price.
Step 2: If $F(0) < 1$ there is no mass point at 0, since then a firm can make a positive profit by charging some positive price.
Step 3: There is no $[p', P'']$ in the support over which F is constant, since a firm at p' can raise its price to p'' without losing any customers.
Step 4: If $F(v) < 1$, a firm charging p, where $0 < p < v$ will earn positive profit while firm charging $p > v$ will earn zero profit. Q.E.D.

Firms' Behavior (II)

If F is continuous (there is no mass points), then for $p \leq v$,

$$H(p) = pm \sum_{n=1}^{\infty} q_n n(1 - F(p))^{n-1} \tag{6}$$

where $m \sum_{n=1}^{\infty} q_n n$ is the number of consumers who sample a firm.

Proposition 1: Given $\{q_i\}_{n=1}^{\infty}$, there exits a unique firm equilibrium (F, H) in which
(i) If $q_1 = 1$, $H = mv$ and F is concentrated on v.
(ii) If $q_1 = 0$, $H = 0$ and F is concentrated on 0.
(iii) If $0 < q_1 < 1$, F is continuous with support $[b, v]$ wehre $b > 0$ and

$$H = mq_1 v = mb \sum_{n=1}^{\infty} nq_n > 0.$$

Firms' Behavior (II')

Proof of Proposition 1:
(i) Since all consumers sample just once, thety accept any price as long as it is below v. Thus all firms set the highest possible price $p = v$.
(ii) First, $p = 0$ is clearly an equilibrium price. A firm looses customers by raising price because all other firms are setting price equals to zero.
By lemma, if there is another equilibrium, it must have continuous F. It follows from (6), when p approaches v the supremum of the support, $F(p)$ approaches 1 and hence $H(p)$ approaches 0 (note $q_1 = 0$). Therefore $H = 0$. However, $H(p) > 0$ for $p > 0$ such that $F(p) < 1$. Contradiction.
(iii) In this case, $p = 0$ for all firms is not an equilibrium. Thus, if (F, H) is an equilibrium, F has tobe continuous and $H = H(p)$ is given by expression (6). First, note that the supremum of the support of F is v. Otherwise a firm charging a price arbitrarily close to v can increase profit by changing its price to v, since most of its buyers are those who have only once price quotation. Thus, $H = H(v) = vmq_1$. Subsitituing this into (6) and rearranging, we get

$$vq_1/p = \sum_{n=1}^{\infty} nq_n(1 - F(p))^{n-1}. \tag{7}$$

Note that $F(\cdot)$ appears only on the right hand side (RHS) of the equation. The RHS is a continuous, monotone increasing function of $(1 - F(p))$, which takes value in $[q_1, \sum_{n=1}^{\infty} nq_n]$. Hence it has a continuous, monotone increasing inverse $T(\cdot)$, which takes the values in $[0, 1]$. Thus,

$$F(p) = 1 - T(vq_1/p) \quad \text{for all } p \text{ in the support}, \tag{8}$$

establishing existence and uniquess.
Finally, since there is no mass at b, $H = H(b) = bm \sum_{n=1}^{\infty} nq_n$, and since $H = vmq_1 > 0$, it follows that $b > 0$.

Definition of an Equilibrium and the Results

Given the price distribution $F(p)$, the cost of purchase incurred by a consumer who takes n observations is

$$e(n) = c \cdot (n-1) + \int_b^v np(1-F(p))^{n-1} dF(p). \qquad (9)$$

Definition: $(F, H, \{q_i\}_{i=1}^{\infty})$ is an equilibrium if

(a) (F, H) is a firm equilibrium given $\{q_i\}_{i=1}^{\infty}$;

(b) for each i such that q_i, i minimizes $e(n)$.

Proposition 2:

(i) There always exists and equilibrium with $p = v$.

(ii) There is no equilibrium with $p = 0$.

Proposition 3:

There exists a number C such that

there exists two dispersed price equilibrium if $c < C$,

there exists one dispersed price equilibrium if $c = C$, and

there is no dispersed price equilibrium if $c > C$.

Proof of Proposition 3

Proof of Proposition 3:
Step 1: At any equilibrium, $q_1 + q_2 = 1$ and $q_1 > 0$. Since $e(n)$ is convex, it is minizmized at one or at most two consecutive values of n. From Proposition 1-(ii), if q_1 the equilibrium is concentrated in $p = 0$, but from Proposition 2 there is no such equilibrium. Therefore, $q_1 > 0$ and $q_1 + q_2 = 1$.
Denote $q_1 = q$ and $q_2 = 1 - q$. Let (F_q, H_q) be the unique firm equilibrium given q, and let $b(q)$ denote the lower bound of the support of F_q.
Step 2: We claim that for $0 < q < 1$, $b(q) = vq/2(1-q)$ and $F_q(p) = 1 - \frac{vq}{2p(1-q)}$ for p in $[b, v]$
Why? From proposition 1-(iii), $H_q = mvq = mp[q + 2(1-q)(1 - F_q(p))]$. Solve this to get $F_q(p)$ for all p in the support and solve $F_q(b) = 0$ to get $b(q)$.

Let $D(q)$ be the expected price paid by a consumer with one observation minus the expected price paid by a consumer with two observations, i.e. $D(q)$ is the expected gain for consumer from searching for the second observation.

$$
\begin{aligned}
D(q) &= \int_{b(q)}^{v} pdF_q(p) - \int_{b(q)}^{v} pd[1-(1-F_q(p))^2] = \int_{b(q)}^{v} pdF_q(p) - \int_{b(q)}^{v} pd[2F_q(q) - F_q(p)^2] \\
&= \left([pF_q(p)]_{b(q)}^{v} - \int_{b(q)}^{v} F_q(p)dp\right) - \left([p(2F_q(p) - F_q(p)^2)]_{b(q)}^{v} - \int_{b(q)}^{v} 2F_q(p) - F_q(p)^2 dp\right) \\
&= \left(v - \int_{b(q)}^{v} F_q(p)dp\right) - \left(v - \int_{b(q)}^{v} 2F_q(p) - F_q(p)^2 dp\right) = \int_{b(q)}^{v} F_q(p)dp - \int_{b(q)}^{v} [F_q(p)]^2 dp \\
&= \int_{b(q)}^{v} F_q(p) - F_q(p)^2 dp = \int_{vq/2(1-q)}^{v} \{1 - \frac{vq}{2p(1-q)} - [1 - \frac{vq}{2p(1-q)}]^2\} dp.
\end{aligned}
$$

Step 3: $D(q)$ is 0 when $q = 0$. It increases as q increases and takes the highest value (let this value be C) at some $q < 1$ and then it starts to decrease as q further increases. Finally it reaches 0 again at $q = 1$. The equation $D(q) = c$ has two solutions for q which are in between 0 and 1 if $c < C$, one solution if $c = C$, and no solution if $c > C$. Q.E.D.

Proof of Proposition 3 (Shape of $D(q)$)