余因子展開

戸瀬　信之

ITOSE PROJECT

V01 for SLIN2020 L12 2020/10/23 at Komaba

(i,j) 余因子 (1)

$A = (a_{ij}) \in M_n(\mathbf{K})$ について考えます.
A から i 行, j 列を除いた $(n-1)$ 次正方行列を A_{ij} とする.

$$\tilde{A}_{ij} = (-1)^{i+j} \det(A_{ij})$$

を A の (i,j) 余因子と呼びます.

(i,j) 余因子 (2)

$A = (\vec{a}\ \vec{b}\ \vec{c}\ \vec{d}) \in M_4(\mathbf{K})$ に対して

$$\tilde{A}_{23} = (-1)^{2+3} \left| \begin{array}{cc|c} a_1 & b_1 & d_1 \\ \hline a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{array} \right|$$

$$\tilde{A}_{32} = (-1)^{3+2} \left| \begin{array}{c|cc} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ \hline a_4 & c_4 & d_4 \end{array} \right|$$

j 列の余因子展開 (1)

$A = (a_{ij}) \in M_n(\mathbf{K})$ に対して

$$
\begin{aligned}
|A| &= a_{1j}\tilde{A}_{1j} + a_{2j}\tilde{A}_{2j} + \cdots + a_{nj}\tilde{A}_{nj} \\
&= \begin{pmatrix} \tilde{A}_{1j} & \tilde{A}_{2j} & \cdots & \tilde{A}_{nj} \end{pmatrix} \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}
\end{aligned}
$$

j 列の余因子展開 (2)—余因子行列

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \cdots & \tilde{A}_{i1} & \cdots & \tilde{A}_{n1} \\ \vdots & & \vdots & & \vdots \\ \tilde{A}_{1j} & \cdots & \tilde{A}_{ij} & \cdots & \tilde{A}_{nj} \\ \vdots & & \vdots & & \vdots \\ \tilde{A}_{1n} & \cdots & \tilde{A}_{in} & \cdots & \tilde{A}_{nn} \end{pmatrix}$$

j 列の余因子展開 (3)

$A = (\vec{a}\ \vec{b}\ \vec{c}\ \vec{d}) \in M_4(\mathbf{K})$ の 3 列に関する余因子展開を考える

$$
\begin{aligned}
|A| &= |\vec{a}\ \vec{b}\ c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 + c_4\vec{e}_4\ \vec{d}| \\
&= c_1|\vec{a}\ \vec{b}\ \vec{e}_1\ \vec{d}| + c_2|\vec{a}\ \vec{b}\ \vec{e}_2\ \vec{d}| + c_3|\vec{a}\ \vec{b}\ \vec{e}_3\ \vec{d}| + c_4|\vec{a}\ \vec{b}\ \vec{e}_4\ \vec{d}| \\
&= c_1\begin{vmatrix} a_1 & b_1 & 1 & d_1 \\ a_2 & b_2 & 0 & d_2 \\ a_3 & b_3 & 0 & d_3 \\ a_4 & b_4 & 0 & d_4 \end{vmatrix} + c_2\begin{vmatrix} a_1 & b_1 & 0 & d_1 \\ a_2 & b_2 & 1 & d_2 \\ a_3 & b_3 & 0 & d_3 \\ a_4 & b_4 & 0 & d_4 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 & 0 & d_1 \\ a_2 & b_2 & 0 & d_2 \\ a_3 & b_3 & 1 & d_3 \\ a_4 & b_4 & 0 & d_4 \end{vmatrix} + c_4\begin{vmatrix} a_1 & b_1 & 0 & d_1 \\ a_2 & b_2 & 0 & d_2 \\ a_3 & b_3 & 0 & d_3 \\ a_4 & b_4 & 1 & d_4 \end{vmatrix} \\
&= c_1(-1)^{3-1}\begin{vmatrix} 1 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 0 & a_3 & b_3 & d_3 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix} + c_2(-1)^{3-1}\begin{vmatrix} 0 & a_1 & b_1 & d_1 \\ 1 & a_2 & b_2 & d_2 \\ 0 & a_3 & b_3 & d_3 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix} \\
&\qquad + c_3(-1)^{3-1}\begin{vmatrix} 0 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 1 & a_3 & b_3 & d_3 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix} + c_4(-1)^{3-1}\begin{vmatrix} 0 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 0 & a_3 & b_3 & d_3 \\ 1 & a_4 & b_4 & d_4 \end{vmatrix}
\end{aligned}
$$

j 列の余因子展開 (4)

$$
\begin{aligned}
|A| &= c_1(-1)^{3-1}(-1)^{1-1}\begin{vmatrix} 1 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 0 & a_3 & b_3 & d_3 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix} + c_2(-1)^{3-1}(-1)^{2-1}\begin{vmatrix} 1 & a_2 & b_2 & d_2 \\ 0 & a_1 & b_1 & d_1 \\ 0 & a_3 & b_3 & d_3 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix} \\
&\quad + c_3(-1)^{3-1}(-1)^{3-1}\begin{vmatrix} 1 & a_3 & b_3 & d_3 \\ 0 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix} + c_4(-1)^{3-1}(-1)^{4-1}\begin{vmatrix} 1 & a_4 & b_4 & d_4 \\ 0 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 0 & a_3 & b_3 & d_3 \end{vmatrix} \\
&= c_1(-1)^{1+3}\begin{vmatrix} 1 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 0 & a_3 & b_3 & d_3 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix} + c_2(-1)^{2+3}\begin{vmatrix} 1 & a_2 & b_2 & d_2 \\ 0 & a_1 & b_1 & d_1 \\ 0 & a_3 & b_3 & d_3 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix} \\
&\quad + c_3(-1)^{3+3}\begin{vmatrix} 1 & a_3 & b_3 & d_3 \\ 0 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix} + c_4(-1)^{4+3}\begin{vmatrix} 1 & a_4 & b_4 & d_4 \\ 0 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 0 & a_3 & b_3 & d_3 \end{vmatrix}
\end{aligned}
$$

j 列の余因子展開 (5)

$$
\begin{aligned}
&= c_1(-1)^{1+3}\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} + c_2(-1)^{2+3}\begin{vmatrix} a_1 & b_1 & d_1 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} \\
&\qquad + c_3(-1)^{3+3}\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_4 & b_4 & d_4 \end{vmatrix} + c_4(-1)^{4+3}\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \\
&= c_1\tilde{A}_{13} + c_2\tilde{A}_{23} + c_3\tilde{A}_{33} + c_4\tilde{A}_{43}
\end{aligned}
$$

i 行の余因子展開 (1)

$A = (a_{ij}) \in M_n(\mathbf{K})$ に対して

$$
\begin{aligned}
|A| &= a_{i1}\tilde{A}_{i1} + a_{i2}\tilde{A}_{i2} + \cdots + a_{in}\tilde{A}_{in} \\
&= \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} \tilde{A}_{i1} \\ \tilde{A}_{i2} \\ \vdots \\ \tilde{A}_{in} \end{pmatrix}
\end{aligned}
$$

i 行の余因子展開 (2)—余因子行列

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \cdots & \tilde{A}_{i1} & \cdots & \tilde{A}_{n1} \\ \vdots & & \vdots & & \vdots \\ \tilde{A}_{1j} & \cdots & \tilde{A}_{ij} & \cdots & \tilde{A}_{nj} \\ \vdots & & \vdots & & \vdots \\ \tilde{A}_{1n} & \cdots & \tilde{A}_{in} & \cdots & \tilde{A}_{nn} \end{pmatrix}$$

i 行の余因子展開 (3)

$A = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix}$ に対して 2 行の余因子展開を考えます.

$$
\begin{aligned}
|A| &= \begin{vmatrix} \mathbf{a} \\ b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + b_4\mathbf{e}_4 \\ \mathbf{c} \\ \mathbf{d} \end{vmatrix} \\
&= b_1 \begin{vmatrix} \mathbf{a} \\ \mathbf{e}_1 \\ \mathbf{c} \\ \mathbf{d} \end{vmatrix} + b_2 \begin{vmatrix} \mathbf{a} \\ \mathbf{e}_2 \\ \mathbf{c} \\ \mathbf{d} \end{vmatrix} + b_3 \begin{vmatrix} \mathbf{a} \\ \mathbf{e}_3 \\ \mathbf{c} \\ \mathbf{d} \end{vmatrix} + b_4 \begin{vmatrix} \mathbf{a} \\ \mathbf{e}_4 \\ \mathbf{c} \\ \mathbf{d} \end{vmatrix} \\
&= b_1 \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ 1 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 1 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 1 & 0 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} + b_4 \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 1 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}
\end{aligned}
$$

i 行の余因子展開 (4)

$$
\begin{aligned}
|A| &= b_1(-1)^{2-1}\begin{vmatrix} 1 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} + b_2(-1)^{2-1}\begin{vmatrix} 0 & 1 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \\
&\quad + b_3(-1)^{2-1}\begin{vmatrix} 0 & 0 & 1 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} + b_4(-1)^{2-1}\begin{vmatrix} 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \\
&= b_1(-1)^{2-1}(-1)^{1-1}\begin{vmatrix} 1 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} + b_2(-1)^{2-1}(-1)^{2-1}\begin{vmatrix} 1 & 0 & 0 & 0 \\ a_2 & a_1 & a_3 & a_4 \\ c_2 & c_1 & c_3 & c_4 \\ d_2 & d_1 & d_3 & d_4 \end{vmatrix} \\
&\quad + b_3(-1)^{2-1}(-1)^{3-1}\begin{vmatrix} 1 & 0 & 0 & 0 \\ a_3 & a_1 & a_2 & a_4 \\ c_3 & c_1 & c_2 & c_4 \\ d_3 & d_1 & d_2 & d_4 \end{vmatrix} + b_4(-1)^{2-1}(-1)^{4-1}\begin{vmatrix} 1 & 0 & 0 & 0 \\ a_4 & a_1 & a_2 & a_3 \\ c_4 & c_1 & c_2 & c_3 \\ d_4 & d_1 & d_1 & d_3 \end{vmatrix}
\end{aligned}
$$

i 行の余因子展開 (5)

$$
\begin{aligned}
|A| &= b_1(-1)^{2+1}\left|\begin{array}{ccc} a_2 & a_3 & a_4 \\ \hline c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{array}\right| + b_2(-1)^{2+2}\left|\begin{array}{c|cc} a_1 & a_3 & a_4 \\ \hline c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{array}\right| \\
&\quad + b_3(-1)^{2+3}\left|\begin{array}{cc|c} a_1 & a_2 & a_4 \\ \hline c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{array}\right| + b_4(-1)^{2+4}\left|\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{array}\right| \\
&= b_1\tilde{A}_{21} + b_2\tilde{A}_{22} + b_3\tilde{A}_{23} + b_4\tilde{A}_{24}
\end{aligned}
$$