余因子展開

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ITOSE PROJECT

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(i,j) 余因子 (1)

i行j列.

$(i,j)=(4,2)$

$A=(a_{ij})\in M_n(\mathbf{K})$ について考えます.
A から i 行, j 列を除いた $(n-1)$ 次正方行列を A_{ij} とする.

$$\tilde{A}_{ij}=(-1)^{i+j}\det(A_{ij})$$

を A の (i,j) 余因子と呼びます.

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}$$

$$A_{4,2}=\begin{pmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{pmatrix}$$

3次正方行列

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix} \qquad \begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

(i,j) 余因子 (2)

$A = (\vec{a}\ \vec{b}\ \vec{c}\ \vec{d}) \in M_4(\mathbf{K})$ に対して

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}$$

$$\tilde{A}_{23} = (-1)^{2+3} \begin{vmatrix} a_1 & b_1 & d_1 \\ \hline a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix}$$

$$\tilde{A}_{32} = (-1)^{3+2} \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ \hline a_4 & c_4 & d_4 \end{vmatrix}$$

j 列の余因子展開 (1)

$A = (a_{ij}) \in M_n(\mathbf{K})$ に対して

$$|A| = a_{1j}\tilde{A}_{1j} + a_{2j}\tilde{A}_{2j} + \cdots + a_{nj}\tilde{A}_{nj}$$

$$= \begin{pmatrix} \tilde{A}_{1j} & \tilde{A}_{2j} & \cdots & \tilde{A}_{nj} \end{pmatrix} \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$$

$$\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$$

行

j 列

$\vec{a}_j$

A の j 列.

$\tilde{A}A$ 余因子行列

$$\left| \vec{a}_1 \; \cdots \; \vec{a}_j \; \vec{a}_{j+1} \; \vec{a}_{j+1} \; \cdots \right|$$

$$\parallel$$

$$(-1)^{j-1} \left| \vec{a}_j \; \vec{a}_1 \; \vec{a}_2 \; \cdots \; \vec{a}_{j-1} \; \vec{a}_{j+1} \; \cdots \right|$$

j 3 4

$$\begin{vmatrix} a|_1 \\ \vdots \\ a|_i \\ a|_{i+1} \\ \vdots \end{vmatrix} = (-1)^{i-1} \begin{vmatrix} a|_i \\ a|_1 \\ a|_2 \\ \vdots \\ a|_{i-1} \\ a|_{i+1} \end{vmatrix}$$

i [ ] i [ ]

j 列の余因子展開 (2)—余因子行列

j行 i列 A_{ij}

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \cdots & \tilde{A}_{i1} & \cdots & \tilde{A}_{n1} \\ \vdots & & \vdots & & \vdots \\ \tilde{A}_{1j} & \cdots & \tilde{A}_{ij} & \cdots & \tilde{A}_{nj} \\ \vdots & & \vdots & & \vdots \\ \tilde{A}_{1n} & \cdots & \tilde{A}_{in} & \cdots & \tilde{A}_{nn} \end{pmatrix} \leftarrow j行$$

↑

i列

j 列の余因子展開 (3)

$A = (\vec{a}\ \vec{b}\ \vec{c}\ \vec{d}) \in M_4(\mathbf{K})$ の 3 列に関する余因子展開を考える

$$|A| = |\vec{a}\ \vec{b}\ \boxed{c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 + c_4\vec{e}_4}\ \vec{d}|$$

3列の線形性.

$$= c_1|\vec{a}\ \vec{b}\ \vec{e}_1\ \vec{d}| + c_2|\vec{a}\ \vec{b}\ \vec{e}_2\ \vec{d}| + c_3|\vec{a}\ \vec{b}\ \vec{e}_3\ \vec{d}| + c_4|\vec{a}\ \vec{b}\ \vec{e}_4\ \vec{d}|$$

$$= c_1 \begin{vmatrix} a_1 & b_1 & 1 & d_1 \\ a_2 & b_2 & 0 & d_2 \\ a_3 & b_3 & 0 & d_3 \\ a_4 & b_4 & 0 & d_4 \end{vmatrix} + c_2 \begin{vmatrix} a_1 & b_1 & 0 & d_1 \\ a_2 & b_2 & 1 & d_2 \\ a_3 & b_3 & 0 & d_3 \\ a_4 & b_4 & 0 & d_4 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 & 0 & d_1 \\ a_2 & b_2 & 0 & d_2 \\ a_3 & b_3 & 1 & d_3 \\ a_4 & b_4 & 0 & d_4 \end{vmatrix} + c_4 \begin{vmatrix} a_1 & b_1 & 0 & d_1 \\ a_2 & b_2 & 0 & d_2 \\ a_3 & b_3 & 0 & d_3 \\ a_4 & b_4 & 1 & d_4 \end{vmatrix}$$

$$= c_1(-1)^{3-1} \begin{vmatrix} 1 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 0 & a_3 & b_3 & d_3 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix} + c_2(-1)^{3-1} \begin{vmatrix} 0 & a_1 & b_1 & d_1 \\ 1 & a_2 & b_2 & d_2 \\ 0 & a_3 & b_3 & d_3 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix}$$

$$+ c_3(-1)^{3-1} \begin{vmatrix} 0 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 1 & a_3 & b_3 & d_3 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix} + c_4(-1)^{3-1} \begin{vmatrix} 0 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 0 & a_3 & b_3 & d_3 \\ \boxed{1 & a_4 & b_4 & d_4} \end{vmatrix}$$

$$\begin{array}{c} 3 \\ \downarrow \end{array}$$

$$|\vec{a}\ \vec{b}\ \vec{e}\ \vec{d}|$$

$$= -|\vec{a}\ \vec{e}\ \vec{b}\ \vec{d}|$$

$$= (-1)^{2}|\vec{e}\ \vec{a}\ \vec{b}\ \vec{d}| \quad 3-1.$$

$$\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = c_1 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{=\vec{e}_1} + c_2 \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{=\vec{e}_2} + c_3 \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{=\vec{e}_3} + c_4 \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{=\vec{e}_4}$$

j 列の余因子展開 (4)

$$|A| = c_1(-1)^{3-1}(-1)^{1-1}\begin{vmatrix} 1 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 0 & a_3 & b_3 & d_3 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix} + c_2(-1)^{3-1}(-1)^{2-1}\begin{vmatrix} 1 & a_2 & b_2 & d_2 \\ 0 & a_1 & b_1 & d_1 \\ 0 & a_3 & b_3 & d_3 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix}$$

$$+ c_3(-1)^{3-1}(-1)^{3-1}\begin{vmatrix} 1 & a_3 & b_3 & d_3 \\ 0 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix} + c_4(-1)^{3-1}(-1)^{4-1}\begin{vmatrix} 1 & a_4 & b_4 & d_4 \\ 0 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 0 & a_3 & b_3 & d_3 \end{vmatrix}$$

$$= c_1(-1)^{1+3}\begin{vmatrix} 1 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 0 & a_3 & b_3 & d_3 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix} + c_2(-1)^{2+3}\begin{vmatrix} 1 & a_2 & b_2 & d_2 \\ 0 & a_1 & b_1 & d_1 \\ 0 & a_3 & b_3 & d_3 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix}$$

$$+ c_3(-1)^{3+3}\begin{vmatrix} 1 & a_3 & b_3 & d_3 \\ 0 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 0 & a_4 & b_4 & d_4 \end{vmatrix} + c_4(-1)^{4+3}\begin{vmatrix} 1 & a_4 & b_4 & d_4 \\ 0 & a_1 & b_1 & d_1 \\ 0 & a_2 & b_2 & d_2 \\ 0 & a_3 & b_3 & d_3 \end{vmatrix}$$

$\tilde{A}_{33}$ " A_{33}

B：$(n-1)$次正方行列

$$\left|\begin{array}{c|ccc} 1 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right| = |B|$$

$$\left|\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline * & & & \\ \vdots & & B & \\ * & & & \end{array}\right| = |B|$$

j 列の余因子展開 (5)

$$
\begin{aligned}
&= \underbrace{c_1(-1)^{1+3}\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix}}_{=\tilde{A}_{13}} + \overbrace{c_2(-1)^{2+3}\begin{vmatrix} a_1 & b_1 & d_1 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix}}^{\tilde{A}_{23}} \\
&\qquad + \underbrace{c_3(-1)^{3+3}\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_4 & b_4 & d_4 \end{vmatrix}}_{=\tilde{A}_{33}} + \underbrace{c_4(-1)^{4+3}\boxed{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}}_{=\tilde{A}_{43}.} \\
&= \underbrace{c_1\tilde{A}_{13} + c_2\tilde{A}_{23} + c_3\tilde{A}_{33} + c_4\tilde{A}_{43}}_{\text{3列の余因子展開.}}
\end{aligned}
$$

i 行の余因子展開 (1)

$A = (a_{ij}) \in M_n(\mathbf{K})$ に対して

$$(-1)^{i+2}\det(A_{i2})$$

$$\|$$

$$|A| = a_{i1}\tilde{A}_{i1} + a_{i2}\tilde{A}_{i2} + \cdots + a_{in}\tilde{A}_{in}$$

$$= \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} \tilde{A}_{i1} \\ \tilde{A}_{i2} \\ \vdots \\ \tilde{A}_{in} \end{pmatrix}$$

$$\|$$

$$a_i$$

$$\begin{array}{cc} i行 & \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \\ & \begin{array}{ccc} 1列 & 2列 & n列 \end{array} \end{array}$$

$$= \det(A_{in})\cdot(-1)^{i+n}$$

i 行の余因子展開 (2)—余因子行列

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \cdots & \tilde{A}_{i1} & \cdots & \tilde{A}_{n1} \\ \vdots & & \vdots & & \vdots \\ \tilde{A}_{1j} & \cdots & \tilde{A}_{ij} & \cdots & \tilde{A}_{nj} \\ \vdots & & \vdots & & \vdots \\ \tilde{A}_{1n} & \cdots & \tilde{A}_{in} & \cdots & \tilde{A}_{nn} \end{pmatrix}$$

$\mathbf{b} = (b_1\ b_2\ b_3\ b_4)$
$= b_1(1\,0\,0\,0) + b_2(0\,1\,0\,0) + b_3(0\,0\,1\,0) + b_4(0\,0\,0\,1)$

$A = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix}$ に対して 2 行の余因子展開を考えます.

$$|A| = \begin{vmatrix} \mathbf{a} \\ \underline{b_1\mathbf{e}_1} + \underline{b_2\mathbf{e}_2} + \underline{b_3\mathbf{e}_3} + \underline{b_4\mathbf{e}_4} \\ \mathbf{c} \\ \mathbf{d} \end{vmatrix}$$

2行の線形性.

$$= b_1 \begin{vmatrix} \mathbf{a} \\ \mathbf{e}_1 \\ \mathbf{c} \\ \mathbf{d} \end{vmatrix} + b_2 \begin{vmatrix} \mathbf{a} \\ \mathbf{e}_2 \\ \mathbf{c} \\ \mathbf{d} \end{vmatrix} + b_3 \begin{vmatrix} \mathbf{a} \\ \mathbf{e}_3 \\ \mathbf{c} \\ \mathbf{d} \end{vmatrix} + b_4 \begin{vmatrix} \mathbf{a} \\ \mathbf{e}_4 \\ \mathbf{c} \\ \mathbf{d} \end{vmatrix}$$

$$= b_1 \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ 1 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 1 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 1 & 0 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} + b_4 \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 1 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

i 行の余因子展開 (4)

$$
\begin{aligned}
|A| &= b_1(-1)^{2-1}\begin{vmatrix} 1 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} + b_2(-1)^{2-1}\begin{vmatrix} 0 & 1 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \\
&\quad + b_3(-1)^{2-1}\begin{vmatrix} 0 & 0 & 1 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} + b_4(-1)^{2-1}\begin{vmatrix} 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \\
&= b_1(-1)^{2-1}(-1)^{1-1}\begin{vmatrix} 1 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} + b_2(-1)^{2-1}(-1)^{2-1}\begin{vmatrix} 1 & 0 & 0 & 0 \\ a_2 & a_1 & a_3 & a_4 \\ c_2 & c_1 & c_3 & c_4 \\ d_2 & d_1 & d_3 & d_4 \end{vmatrix} \\
&\quad + b_3(-1)^{2-1}(-1)^{3-1}\begin{vmatrix} 1 & 0 & 0 & 0 \\ a_3 & a_1 & a_2 & a_4 \\ c_3 & c_1 & c_2 & c_4 \\ d_3 & d_1 & d_2 & d_4 \end{vmatrix} + b_4\underbrace{(-1)^{2-1}(-1)^{4-1}}_{=(-1)^{2+4}}\begin{vmatrix} 1 & 0 & 0 & 0 \\ a_4 & a_1 & a_2 & a_3 \\ c_4 & c_1 & c_2 & c_3 \\ d_4 & d_1 & d_1 & d_3 \end{vmatrix}
\end{aligned}
$$

i 行の余因子展開 (5)

$$
\begin{aligned}
|A| &= b_1\overbrace{(-1)^{2+1}\begin{vmatrix} a_2 & a_3 & a_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix}}^{\tilde{A}_{21}} + b_2\overbrace{(-1)^{2+2}\begin{vmatrix} a_1 & a_3 & a_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix}}^{\tilde{A}_{22}} \\
&\quad + b_3\underbrace{(-1)^{2+3}\begin{vmatrix} a_1 & a_2 & a_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix}}_{\tilde{A}_{23}} + b_4\underbrace{(-1)^{2+4}\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}}_{\tilde{A}_{24}} \\
&= b_1\tilde{A}_{21} + b_2\tilde{A}_{22} + b_3\tilde{A}_{23} + b_4\tilde{A}_{24}
\end{aligned}
$$