L10

(1) $I_1 = \int_0^2 \frac{1}{x^2+4}\,dx$ $x = 2\tan\theta$ のとき $\begin{array}{c|c} \theta & 0 \nearrow \frac{\pi}{2} \\ \hline x & 0 \nearrow 2 \end{array}$ となる.

$$\frac{dx}{d\theta} = 2\,\frac{1}{\cos^2\theta} = 2\left(1+\tan^2\theta\right)$$

$$x^2+4 = 4\tan^2\theta + 4 = 4(1+\tan^2\theta)$$

から

$$I_1 = \int_0^{\frac{\pi}{2}} \frac{1}{2}\cdot d\theta = \frac{\pi}{4}$$

(2) $I_2 = \int_0^{\frac{3}{2}} \frac{1}{\sqrt{9-x^2}}\,dx$ $x = 3\sin\theta$ のとき $\begin{array}{c|c} \theta & 0 \nearrow \frac{\pi}{6} \\ \hline x & 0 \nearrow \frac{3}{2} \end{array}$

となる $\frac{dx}{d\theta} = 3\cos\theta$

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = 3\sqrt{1-\sin^2\theta} = 3\cos\theta$$

$\left(0 \leq \theta \leq \frac{\pi}{6}\right.$ のとき $\left.\cos\theta > 0\right)$

から

$$I_2 = \int_0^{\frac{\pi}{6}} d\theta = \frac{\pi}{6}$$

(4) $I_4 = \int_1^3 \frac{1}{x\sqrt{x+1}}\,dx.$ $\sqrt{x+1} = t$ となるように $x = t^2 - 1$

とすると $\begin{array}{c|c} t & \sqrt{2} \nearrow 2 \\ \hline x & 1 \nearrow 3 \end{array}$ となる. また $\frac{dx}{dt} = 2t$ となる. よって

$$I_3 = \int_{\sqrt{2}}^2 \frac{2t}{(t^2-1)t}\,dt = 2\int_{\sqrt{2}}^2 \frac{dt}{t^2-1}$$

$$= \int_{\sqrt{2}}^2 \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt$$

$$= \Big[\log(t-1) - \log(t+1)\Big]_{\sqrt{2}}^2$$

$$= (\log 1 - \log 3) - \left(\log(\sqrt{2}-1) - \log(\sqrt{2}+1)\right) = \log\frac{(\sqrt{2}+1)^2}{3}$$

(3) $I_3 \int_0^3 \frac{dx}{(x^2+9)^2}$ $x = 3\tan\theta$ とおくと $\begin{array}{c|c} \theta & 0 \nearrow \frac{\pi}{2} \\ \hline x & 0 \nearrow 3 \end{array}$ となる.

$$\frac{dx}{d\theta} = 3\frac{1}{\cos^2\theta}, \quad x^2+9 = 9(\tan^2\theta+1) = \frac{9}{\cos^2\theta}$$

から

$$I_3 = \int_0^{\frac{\pi}{2}} \frac{3\frac{1}{\cos^2\theta}}{\left(9\cdot\frac{1}{\cos^2\theta}\right)^2} d\theta = \frac{1}{27}\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

$$= \frac{1}{27}\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{54}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{54}\cdot\frac{\pi}{2} = \frac{\pi}{108}$$

(5) $I_5 = \int_0^1 \frac{x^2}{\sqrt{x+1}} dx$ $x+1 = t$ となるように $x = t-1$ とすると

$\begin{array}{c|c} t & 1 \nearrow 2 \\ \hline x & 0 \nearrow 1 \end{array}$ となります. $\frac{dx}{dt} = 1$ から

$$I_5 = \int_1^2 \frac{(t-1)^2}{\sqrt{t}} dt = \int_1^2 \left(t\sqrt{t} - 2\sqrt{t} + \frac{1}{\sqrt{t}}\right) dt$$

$$= \left[\frac{2}{5}t^2\sqrt{t}\right]_1^2 - 2\left[\frac{2}{3}t\sqrt{t}\right]_1^2 + \left[2\sqrt{t}\right]_1^2$$

$$= \frac{2}{5}(4\sqrt{2}-1) - \frac{4}{3}(2\sqrt{2}-1) + 2(\sqrt{2}-1)$$

$$= \frac{14}{15}\sqrt{2} - \frac{16}{15}$$