積分 IV

置換積分（その１）

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ITOSE PROJECT

V01b Nov 24, 2020 for CalcNT

置換積分 (1)

$x.$

開区間 (A_0, B_0) 上の連続関数 $f: (A_0, B_0) \to \mathbf{R}$ とその原始関数 $F(t)$:

$$F'(x) = f(x) \quad (x \in (A_0, B_0))$$

さらに~~連続関数~~ $\varphi: (a_0, b_0) \to \mathbf{R}$ が

微分可能.

$$\varphi(t) \in (A_0, B_0) \quad (t \in (a_0, b_0))$$

→ $f(\varphi(t))$ が定義できる.

が成立するとします．このとき

$$\frac{d}{dt}(F(\varphi(t))) = F'(\varphi(t)) \cdot \varphi'(t) = f(\varphi(t))\varphi'(t)$$

$F(\varphi(t))$ が $f(\varphi(t)) \cdot \varphi'(t)$ の原始関数

置換積分 (2)

$a_0 \quad a \quad t \quad b_0$

$$\int_a^b f(\varphi(t))\varphi'(t)dt = [F(\varphi(t))]_a^b$$

$$= F(\varphi(b)) - F(\varphi(a))$$

$B = \varphi(b)$, $A = \varphi(a)$ とおく

$$= F(B) - F(A) = \int_A^B f(x)dx$$

fの原始関数がF.

公式

$$\int_a^b f(\varphi(t))\varphi'(t)dt = \int_A^B f(x)dx$$

$(1+t^2)' = 2t \longrightarrow \left\{\frac{1}{2}(1+t^2)\right\}' = t$

$$I_1 := \int_0^1 (1+t^2)^4 t\,dt = \frac{1}{2}\int_0^1 \underbrace{(1+t^2)^4}_{(g(t))^4}\underbrace{(1+t^2)'}_{=g'(t)}dt$$

$f(x) = x^4$

$A = g(a)$, $B = g(b)$

$1 = g(0)$, $2 = g(1)$

$$= \frac{1}{2}\int_{1\,(A)}^{2\,(B)} x^4 dx$$

$$= \frac{1}{2}\left[\frac{x^5}{5}\right]_1^2 = \frac{1}{10}(2^5 - 1) = \frac{31}{10}$$

$(x^5)' = 5x^4$

$\left(\frac{1}{5}x^5\right)' = x^4$

ここで対応 $\boxed{\begin{array}{c|ccc} t & 0 & \nearrow & 1 \\ x & 1 & \nearrow & 2 \end{array}}$ を用いています.

$x = \varphi(t) = 1 + t^2$

$\varphi'(t) = 2t$

$$\frac{1}{2}\int_0^1 \frac{2t}{1+t^2}\,dt.$$

$$\int_0^1 \frac{t}{1+t^2}\,dt = \frac{1}{2}\int_0^1 \frac{(1+t^2)'}{1+t^2}\,dt = \frac{1}{2}\int_1^2 \frac{1}{x}\,dx = \frac{1}{2}\left[\log x\right]_1^2 = \frac{1}{2}\log 2$$

$B = \varphi(b)$, $A = \varphi(a)$

$2 = \varphi(1)$ $\quad 0 = \varphi(0)$

$(\log x)' = \dfrac{1}{x}$

$\boxed{\log 1 = 0}$

ここで対応 $\begin{array}{c|ccc} t & 0 & \nearrow & 1 \\ x & 1 & \nearrow & 2 \end{array}$ を用いています.

具体例 (2)

$$\begin{aligned}\int_0^1 \frac{t}{1+t^2}dt &= \frac{1}{2}\int_0^1 \frac{(1+t^2)'}{1+t^2}dt \\ &= \frac{1}{2}\int_1^2 \frac{1}{x}dx \\ &= \frac{1}{2}[\log x]_1^2 = \frac{1}{2}\log 2\end{aligned}$$

ここで対応 $\begin{array}{c|ccc} t & 0 & \nearrow & 1 \\ x & 1 & \nearrow & 2 \end{array}$ を用いています.