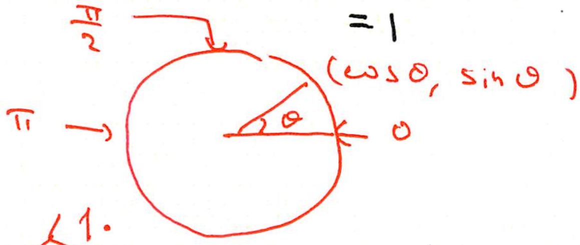
$(\cos t)' = -\sin t$

$(\sin t)' = \cos t$

(1) $(\cos t)' = -\sin t$ なので $\boxed{(-\cos t)' = \sin t}$. 従って

$$\int_0^{\frac{\pi}{2}} \sin t \, dt = \left[-\cos t\right]_0^{\frac{\pi}{2}} = -\left(\cos\frac{\pi}{2} - \cos 0\right) = -(0-1)$$
$$= 1$$



(2) $$\int_0^{\pi} t \sin t \, dt = \int_0^{\pi} t\,(-\cos t)' \, dt$$

$$= -\left[t \cos t\right]_0^{\pi} + \int_0^{\pi} \cos t \, dt$$
$$= -\pi \cdot (-1) + \cdots$$
$$= \pi + \left[\sin t\right]_0^{\pi} = \pi.$$

(3) $$\int_1^{e^2} t \log t \, dt = \int_1^{e^2} \left(\frac{t^2}{2}\right)' \log t \, dt \quad (\log t)'$$

$\log 1 = 0$

$\log e = 1$

$\log e^2 = 2.$

$$= \left[\frac{t^2}{2}\log t\right]_1^{e^2} - \int_1^{e^2} \frac{t^2}{2} \cdot \frac{1}{t} \, dt$$
$$= \frac{1}{2}(e^4 \cdot 2 - 1 \cdot 0)$$
$$= e^4 - \frac{1}{2}\int_1^{e^2} t \, dt$$
$$= e^4 - \frac{1}{2}\left[\frac{1}{2}t^2\right]_1^{e^2}$$
$$= e^4 - \frac{1}{4}(e^4 - 1) = \frac{3e^4 + 1}{4}$$

$(e^{ct})' = c\,e^{ct}$ (c:定数)

$(e^{-t})' = -e^{-t}$

$(-e^{-t})' = e^{-t}$

(4) $$\int_0^1 t e^{-t} \, dt = \int_0^1 t\,(-e^{-t})' \, dt$$

$\boxed{e^0 = 1}$

$$= -\left[t e^{-t}\right]_0^1 + \int_0^1 e^{-t} \, dt$$
$$= -\frac{1}{e} + \left[-e^{-t}\right]_0^1$$
$$= -\frac{1}{e} - \left(\frac{1}{e} - 1\right) = 1 - \frac{2}{e}$$

$\{(x+3)^5\}' = 5(x+3)^4 \cdot 1$

$\{(x+3)^4\}' = 4 \cdot (x+3)^3$ $\nearrow$ $\{\frac{1}{4}(x+3)^4\}'$

$= (x+3)^3$

(5) $$\int_0^1 x(x+3)^3\,dx = \int_0^1 x\left\{\frac{(x+3)^4}{4}\right\}' dx$$

$$= \left[x \cdot \frac{(x+3)^4}{4}\right]_0^1 - \frac{1}{4}\int_0^1 \underbrace{1}\,(x+3)^4\,dx$$

$$= \frac{4^4}{4} - \frac{1}{\cancel{20}}\overset{\frac{1}{4}}{}\left[\frac{1}{5}(x+3)^5\right]_0^1$$

注 $4^5 = 2^{10} = 1024$
$3^5 = 3 \cdot 81 = 243$

$$= 64 - \frac{1}{20}(4^5 - 3^5) = 64 - \frac{1}{20}(1024 - 243)$$

$$= \frac{499}{20}$$

$= \int_0^1 (x+3-3)(x+3)^3\,dx = \int_0^1 (x+3)^4\,dx - \int_0^1 (x+3)^3\,dx$

$= \cdots$

(6) $$\int_0^1 \frac{dt}{4t+1} = \frac{1}{4}\int_0^1 \frac{\boxed{(4t+1)'}}{4t+1}\,dt$$

$x = 4t+1$ とすると

t	$0 \nearrow 1$
x	$1 \nearrow 5$

$f(x) = \frac{1}{x}$

$g(t) = 4t+1$

$$= \int_1^5 \frac{1}{x}\,dx = [\log x]_1^5 = \log 5$$

(別解) $(\log(4t+1))' = \frac{4}{4t+1}$ から $\left(\frac{1}{4}\log(4t+1)\right)' = \frac{1}{4t+1}$

から

$$\int_0^1 \frac{dt}{4t+1} = \left[\frac{1}{4}\log(4t+1)\right]_0^1 = \frac{1}{4}(\log 5 - \log 1)$$

$$= \frac{1}{4}\log 5$$

$f(x) = \dfrac{1}{x^3}$, $\varphi(t) = 2t+1$

(7) $\displaystyle\int_0^1 \frac{dt}{(2t+1)^3} = \frac{1}{2}\int_0^1 \frac{\boxed{(2t+1)'}}{(2t+1)^3}\,dt = \frac{1}{2}\int_1^3 \boxed{\frac{1}{x^3}}\,dx.$

$(2t+1)' = 2$

$x = 2t+1$ とすると

t	$0 \nearrow 1$
x	$1 \nearrow 3$

$\displaystyle = \frac{1}{2}\left[-\frac{1}{2}\frac{1}{x^2}\right]_1^3$

$\left(\dfrac{1}{x^2}\right)' = -\dfrac{2}{x^3}$ → $\left(-\dfrac{1}{2}\cdot\dfrac{1}{x^2}\right)' = \dfrac{1}{x^3}$

$\displaystyle = -\frac{1}{4}\left(\frac{1}{9} - 1\right) = \frac{2}{9}$

(別解)

$\displaystyle \left\{\frac{1}{(2t+1)^2}\right\}' = -2\cdot\frac{2}{(2t+1)^3} = -4\,\frac{1}{(2t+1)^3}$ から

$(2t+1)'$

$\displaystyle \left\{-\frac{1}{4}(2t+1)^{-2}\right\}' = \frac{1}{(2t+1)^3}$ なので

$\displaystyle \int_0^1 \frac{dt}{(2t+1)^3} = \left[-\frac{1}{4}\cdot\frac{1}{(2t+1)^2}\right]_0^1 = -\frac{1}{4}\left(\frac{1}{9} - 1\right)$

(8)

$\displaystyle I := \int_0^1 (1+t^2)^3\, t\,dt = \frac{1}{2}\int_0^1 (1+t^2)^3 \underbrace{(1+t^2)'}_{=2t} dt$

$x = 1+t^2$ とすると

t	$0 \nearrow 1$
x	$1 \nearrow 2$

$f(x) = x^3$

$x = \varphi(t) = 1+t^2$

$\displaystyle I = \frac{1}{2}\int_1^2 x^3\,dx = \frac{1}{2}\left[\frac{1}{4}x^4\right]_1^2 = \frac{1}{8}(16-1) = \frac{15}{8}$

(別解)

$\{(1+t^2)^4\}' = 4(1+t^2)^3\cdot 2t$ から

$\left\{\dfrac{1}{\cancel{2}\,8}(1+t^2)^4\right\}' = (1+t^2)t$

となるので

$\displaystyle I = \left[\frac{1}{\cancel{2}\,8}(1+t^2)^4\right]_0^1 = \frac{1}{\cancel{2}\,8}\cdot(2^4 - 1) = \cdots$

(7) の補足

$a \neq 0$

$$\int_p^q (at+b)^\alpha dt = \frac{1}{a}\int_p^q (at+b)^\alpha \cdot (at+b)' dt$$

$$= \frac{1}{a}\int_B^A x^\alpha dx.$$

但し　$A = ap+b$, $B = aq+b$

(9)

$$\int_0^{\sqrt{3}} \sqrt{1+t^2}\cdot t\,dt = \frac{1}{2}\int_0^{\sqrt{3}} \sqrt{1+t^2}\,(1+t^2)'\,dt$$

$$= \frac{1}{2}\int_1^4 \sqrt{x}\,dx$$

$$= \frac{1}{2}\left[\frac{2}{3}x\sqrt{x}\right]_1^4$$

$$= \frac{1}{3}(4\cdot 2-1) = \frac{7}{3}$$

$(x\sqrt{x})' = (x^{\frac{3}{2}})' = \frac{3}{2}x^{\frac{1}{2}}$

$(\frac{2}{3}x\sqrt{x})' = \sqrt{x}$.

t	$0 \nearrow \sqrt{3}$
x	$1 \nearrow 4$

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$$\int_{-2}^{1} \frac{x}{\sqrt{x+3}}\,dx = \int_{1}^{4} \frac{t-3}{\sqrt{t}} \cdot \underbrace{1}_{=\varphi'(t)}\,dt$$

$x = \varphi(t)$ を $x+3 = t$ となるように作る　$x = t-3 = \varphi(t)$.

$\uparrow$ $\uparrow$

-2 1

1 4

$\varphi'(t) = 1$

t	$1 \nearrow 4$
x	$-2 \nearrow 1$

$$\boxed{dx = \varphi'(t)\,dt}$$

$$= \int_1^4 \left(\sqrt{t} - \frac{3}{\sqrt{t}}\right) dt$$

$(t\sqrt{t})' = \frac{3}{2}\sqrt{t}$

$(\frac{2}{3}t\sqrt{t})' = \sqrt{t}$

$(t\sqrt{t})' = \frac{3}{2}\sqrt{t} \rightsquigarrow (\frac{2}{3}t\sqrt{t})' = \sqrt{t}$

$$= \left[\frac{2}{3}t\sqrt{t} - 3 \cdot 2\sqrt{t}\right]_1^4$$

$(\sqrt{t})' = \frac{1}{2}\cancel{\sqrt{t}}\ \cdot \frac{1}{\sqrt{t}}$　　$(2\sqrt{t})' = \frac{1}{\sqrt{t}}$

$$= \frac{2}{3}(8-1) - 6(2-1)$$

$$= \frac{14}{3} - 6 = -\frac{4}{3}.$$

$$\int_{-2}^{1} \frac{x}{\sqrt{x+3}}\,dx = \int_{1}^{2} \frac{t^2-3}{t}\, 2t\, dt$$

$$2t = (t^2-3)'$$

(10)

$t = \sqrt{x+3}$ とおく ように $x = \varphi(t)$ と表す.

$t^2 = x+3 \rightsquigarrow \quad x = t^2 - 3, \quad t \geq 0$

$t \geq 0$

$t^2 = 4 \longrightarrow t = 2 \quad t \geq 0.$

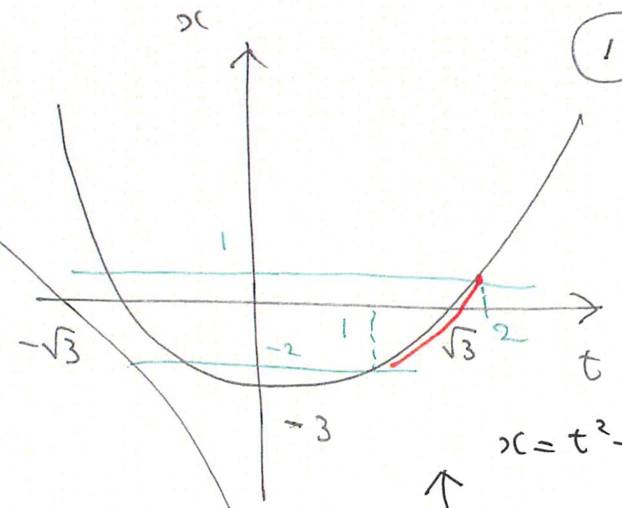
$x = -2$

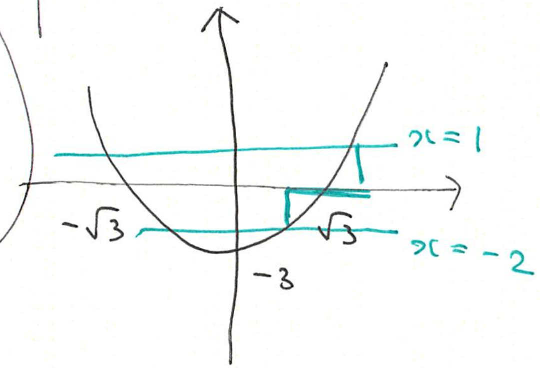
$t^2 = 1$

$\longrightarrow t = 1$

t	1	$\nearrow$	2
x	-2	$\nearrow$	1

$\varphi(t) = t^2 - 3, \quad \varphi'(t) = 2t \qquad dx = 2t\,dt$





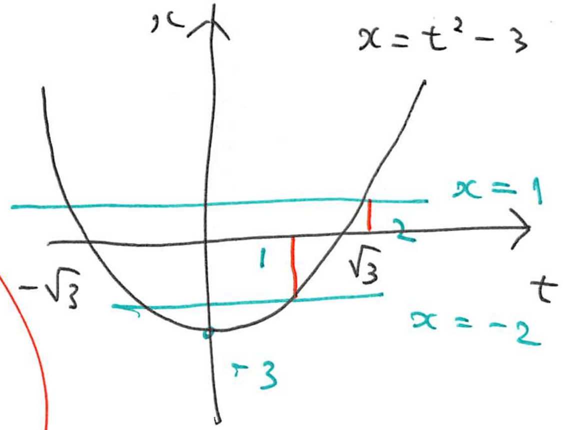
$$= 2\int_1^2 (t^2-3)\,dt$$

$$= 2\left[\frac{t^3}{3} - 3t\right]_1^2 = \cdots$$

$$\int_{-2}^{1} \frac{x}{\sqrt{x+3}}\,dx = \int_{1}^{2} \frac{t^2-3}{t}\cdot 2t\,dt$$

$$x = t^2-3 \longrightarrow \varphi'(t) = 2t$$

$t \geq 0$ "$\varphi(t)$"

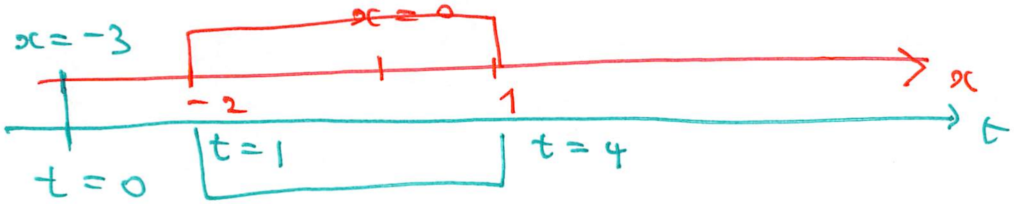


$x = 1 \quad t^2 = 4 \xrightarrow{t \geq 0} t = 2$

$x = -2 \quad t^2 = 1 \longrightarrow t = 1$

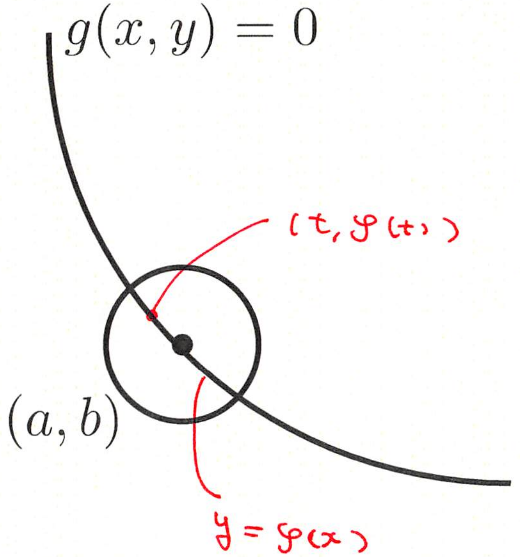
t	$1 \nearrow 2$	
x	$-2 \nearrow 1$	

$$= 2\int_{1}^{2} (t^2-3)\,dt = \ldots$$



$x + 3 = t$

極大・極小の十分条件



$$g(a, b) = 0, \quad g_y(a, b) \neq 0$$

を仮定して, 陰関数定理を適用する. (a, b) の近くで

$$y = \varphi(x)$$

と曲線 $g(x, y) = 0$ を表す.

(a, b) で極大（極小）ならば

$$F(t) = f(t, \varphi(t))$$

とすると $F'(a) = 0$ が従う.

$$F''(a) > 0 \quad (\text{resp.} \quad F''(a) < 0$$

ならば (a, b) で極小（resp. 極大）となります.

解法 (3)

Chain Rule　$G(t) = g(x(t), y(t))$

$G'(t) = g_x(\quad)\cdot x'(t) + g_y(\quad)\cdot y'(t).$

さらに $g(t, \varphi(t)) \equiv 0$ の両辺を t で微分して

$$g_x(t, \varphi(t)) \cdot 1 + g_y(t, \varphi(t)) \cdot \varphi'(t) \equiv 0$$

$\varphi(a) = b$

$t = a$ を代入　$g_x(a,b) + \overbrace{g_y(a,b)}^{\neq 0} \varphi'(a) = 0 \longrightarrow \varphi'(a) = -\dfrac{g_x(a,b)}{g_y(a,b)}$

$$g_{xx}(t, \varphi(t)) + 2g_{xy}(t, \varphi(t)) \cdot \varphi'(t) + g_{yy}(t, \varphi(t)) \cdot \varphi'(t)^2$$
$$+ \; g_y(t, \varphi(t)) \cdot \varphi''(t) \equiv 0$$

を得ます.　もう1回 t で微分.

解法 (4)

$\varphi(a)=b$ $t=a$ を代入

$g_{xx}(a,b)+2g_{xy}(a,b)\cdot\varphi'(a)+g_{yy}(a,b)\cdot\varphi'(a)^2+$

$\underbrace{g_y(a,b)}_{\neq 0}\,\varphi''(a)=0$

$t=a$ とするとき $P_0(a,b)$ と定めて

$$\varphi'(a)=-\frac{g_x(P_0)}{g_y(P_0)}$$

$$\varphi''(a)=-\frac{1}{g_y(a,b)}\left(g_{xx}(P_0)+2g_{xy}(P_0)\cdot\varphi'(a)+g_{yy}(P_0)\cdot\varphi'(a)^2\right)$$

となります.

$\varphi''(a)$

$-\dfrac{g_x(P_0)}{g_y(P_0)}$ $\qquad$ $\dfrac{g_x(P_0)^2}{g_y(P_0)^2}$

$$\varphi''(a) = -\frac{1}{g_y(a,b)}\left(g_{xx}(P_0) + 2g_{xy}(P_0)\cdot\varphi'(a) + g_{yy}(P_0)\cdot\varphi'(a)^2\right)$$

$$= -\frac{1}{g_y(a,b)^3}\Big(g_{xx}(P_0)\cdot g_y(P_0)^2 - 2g_{xy}(P_0)\cdot g_x(P_0)g_y(P_0)$$

$$+ g_{yy}(P_0)\cdot g_x(P_0)^2\Big)$$

$$= \frac{1}{g_y(a,b)^3}\begin{vmatrix} 0 & g_x(a,b) & g_y(a,b) \\ g_x(a,b) & g_{xx}(a,b) & g_{xy}(a,b) \\ g_y(a,b) & g_{yx}(a,b) & g_{yy}(a,b) \end{vmatrix}$$

$+$ $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

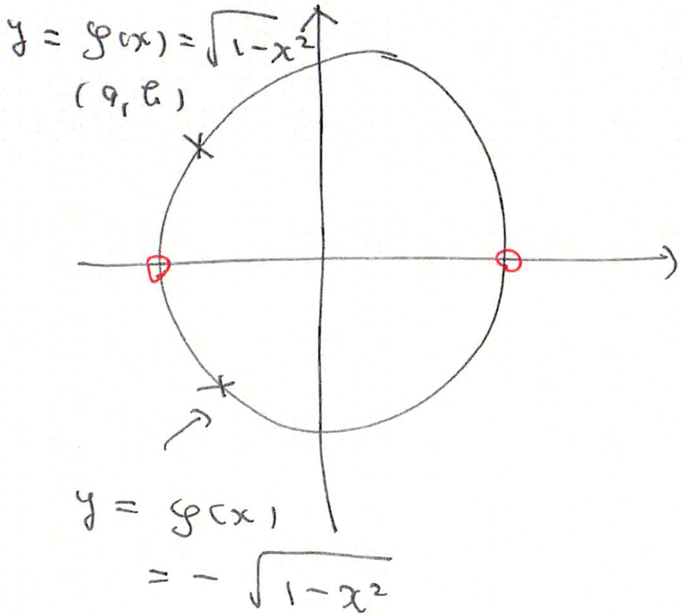
$-$ $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

3,3行目 Hesse行列式

$$\begin{vmatrix} 0 & p & q \\ p & A & C \\ q & C & B \end{vmatrix} = -Aq^2 + 2Cpq - Bp^2$$

(3)

$$g(x, y) = x^2 + y^2 - 1 = 0$$



$$g_y = 2y$$

$$g_y(a, b) = 2b \neq 0$$

$$g_x = 2x, \quad g_y = 2y$$

$$g_{xx} = 2, \quad g_{xy} = g_{yx} = 0, \quad g_{yy} = 2$$

$$\varphi''(a) = \frac{1}{(2b)^3} \begin{vmatrix} 0 & 2a & 2b \\ 2a & 2 & 0 \\ 2b & 0 & 2 \end{vmatrix}$$

$a^2 + b^2 = 1$

$$= \frac{1}{8b^3}\left(-8b^2 - 8a^2\right) = -\frac{1}{b^3}$$

I (1) $\displaystyle\int_1^e \frac{\log x}{\sqrt{x}}\,dx$

(2) $\displaystyle\int_0^1 x^2(x-1)^2\,dx$ 部分積分 2回

(3) $\displaystyle\int_0^1 \frac{1}{\sqrt{2x+1}}\,dx$

(4) $\displaystyle\int_0^1 t\,e^{-\frac{1}{2}t^2}\,dt \qquad \left(e^{-\frac{1}{2}t^2}\right)' = ?$

(5) $\displaystyle\int_0^1 x\sqrt{3x+1}\,dx$

$3x+1=t$ とおき $x=\varphi(t)$ を求める.

(6) $\displaystyle\int_0^1 \frac{x-1}{(x-2)^3}\,dx$

$t=x-2$ とおくならば $x=\varphi(t)$ を求める.

II $g(x,y)=x^2+\dfrac{y^2}{4}-1=0 \qquad y=2\sqrt{1-x^2}=\varphi(x)$

$(0,2)$ における $\varphi''(0)$ を求める.