(1) $(\cos t)' = -\sin t$ なので $(-\cos t)' = \sin t$. 従って

$$\int_0^{\frac{\pi}{2}} \sin t\, dt = \Big[-\cos t\Big]_0^{\frac{\pi}{2}} = -\left(\cos\frac{\pi}{2} - \cos 0\right) = -(0-1)$$
$$= 1$$

(2)
$$\int_0^{\pi} t\sin t\, dt = \int_0^{\pi} t(-\cos t)'\, dt$$
$$= -\Big[t\cos t\Big]_0^{\pi} + \int_0^{\pi} \cos t\, dt$$
$$= \pi + \Big[\sin t\Big]_0^{\pi} = \pi.$$

(3)
$$\int_1^{e^2} t\log t\, dt = \int_1^{e^2} \left(\frac{t^2}{2}\right)' \log t\, dt$$
$$= \left[\frac{t^2}{2}\log t\right]_1^{e^2} - \int_1^{e^2} \frac{t^2}{2}\cdot\frac{1}{t}\, dt$$
$$= e^4 - \frac{1}{2}\int_1^{e^2} t\, dt$$
$$= e^4 - \frac{1}{2}\left[\frac{1}{2}t^2\right]_1^{e^2}$$
$$= e^4 - \frac{1}{4}(e^4 - 1) = \frac{3e^4+1}{4}$$

(4)
$$\int_0^1 t e^{-t}\, dt = \int_0^1 t(-e^{-t})'\, dt$$
$$= -\Big[t e^{-t}\Big]_0^1 + \int_0^1 e^{-t}\, dt$$
$$= -\frac{1}{e} + \Big[-e^{-t}\Big]_0^1$$
$$= -\frac{1}{e} - \left(\frac{1}{e} - 1\right) = 1 - \frac{2}{e}$$

(5) $\displaystyle\int_0^1 x(x+3)^3\,dx = \int_0^1 x\left\{\frac{(x+3)^4}{4}\right\}'dx$

$\displaystyle = \left[x\cdot\frac{(x+3)^4}{4}\right]_0^1 - \frac{1}{4}\int_0^1 (x+3)^4\,dx$

$\displaystyle = \frac{4^4}{4} - \frac{1}{20}\left[(x+3)^5\right]_0^1$

$\boxed{\text{注}\quad 4^5 = 2^{10} = 1024 \quad 3^5 = 3\cdot 81 = 243}$

$\displaystyle = 64 - \frac{1}{20}(4^5 - 3^5) = 64 - \frac{1}{20}(1024 - 243)$

$\displaystyle = \frac{499}{20}$

(6) $\displaystyle\int_0^1 \frac{dt}{4t+1} = \frac{1}{4}\int_0^1 \frac{(4t+1)'}{4t+1}\,dt$

$x = 4t+1$ とすると

t	$0 \nearrow 1$
x	$1 \nearrow 5$

$\displaystyle = \frac{1}{4}\int_1^5 \frac{1}{x}\,dx = \frac{1}{4}\left[\log x\right]_1^5 = \frac{\log 5}{4}$

(別解) $\displaystyle(\log(4t+1))' = \frac{4}{4t+1}$ から $\displaystyle\left(\frac{1}{4}\log(4t+1)\right)' = \frac{1}{4t+1}$

から

$\displaystyle\int_0^1 \frac{dt}{4t+1} = \left[\frac{1}{4}\log(4t+1)\right]_0^1 = \frac{1}{4}(\log 5 - \log 1)$

$\displaystyle = \frac{1}{4}\log 5$

(5)の別解.

$\displaystyle\int_0^1 x(x+3)^3\,dx = \int_0^1 (x+3)^4\,dx - 3\int_0^1 (x+3)^3\,dx$

$\displaystyle = \left[\frac{1}{5}(x+3)^5\right]_0^1 - 3\left[\frac{1}{4}(x+3)^4\right]_0^1$

$= \cdots$

(7)

$$\int_0^1 \frac{dt}{(2t+1)^3} = \frac{1}{2}\int_0^1 \frac{(2t+1)'}{(2t+1)^3}\,dt = \frac{1}{2}\int_1^3 \frac{1}{x^3}$$

$x = 2t+1$ とすると

t	$0 \nearrow 1$
x	$1 \nearrow 3$

$$= \frac{1}{2}\left[-\frac{1}{2}\frac{1}{x^2}\right]_1^3$$

$$= -\frac{1}{4}\left(\frac{1}{9}-1\right) = \frac{2}{9}$$

(別解)

$$\left\{\frac{1}{(2t+1)^2}\right\}' = -2\cdot\frac{2}{(2t+1)^3} = -4\,\frac{1}{(2t+1)^3} \text{ から}$$

$$\left\{-\frac{1}{4}(2t+1)^{-2}\right\}' = \frac{1}{(2t+1)^3} \text{ なので}$$

$$\int_0^1 \frac{dt}{(2t+1)^3} = \left[-\frac{1}{4}\cdot\frac{1}{(2t+1)^2}\right]_0^1 = -\frac{1}{4}\left(\frac{1}{9}-1\right)$$

(8)

$$I := \int_0^1 (1+t^2)^3\, t\, dt = \frac{1}{2}\int_0^1 (1+t^2)^3 (1+t^2)'\,dt$$

$x = 1+t^2$ とすると

t	$0 \nearrow 1$
x	$1 \nearrow 2$

$$I = \frac{1}{2}\int_1^2 x^3\,dx = \frac{1}{2}\left[\frac{1}{4}x^4\right]_1^2 = \frac{1}{8}(16-1) = \frac{15}{8}$$

(別解)

$$\{(1+t^2)^4\}' = 4(1+t^2)^3\cdot 2t \text{ から}$$

$$\left\{\frac{1}{\cancel{2}\ 8}(1+t^2)^4\right\}' = (1+t^2)\,t$$

となるので

$$I = \left[\frac{1}{\cancel{2}\ 8}(1+t^2)^4\right]_0^1 = \frac{1}{\cancel{2}\ 8}\cdot(2^4-1) = \cdots$$

(9)

$$\int_0^{\sqrt{3}} \sqrt{1+t^2} \cdot t\,dt = \frac{1}{2}\int_0^{\sqrt{3}} \sqrt{1+t^2}\,(1+t^2)'\,dt$$

$$= \frac{1}{2} \int_1^4 \sqrt{x}\,dx$$

$$= \frac{1}{2}\left[\frac{2}{3}x\sqrt{x}\right]_1^4$$

$$= \frac{1}{3}(4\cdot 2-1) = \frac{7}{3}$$

t	$0 \nearrow \sqrt{3}$
x	$1 \nearrow 4$