2019 年 12 月 18 日小テスト解答

I $I, p, q > 0$ とします．制約条件（予算制約）

1財　2財

$$g(x,y) := I - px - qy = 0$$

価格　p　q

の下で効用関数

$$u(x,y) := x^{\frac{1}{4}}y^{\frac{1}{4}}$$

購入量　x　y.

を考えます．停留点を求めましょう．

$u_x = \frac{1}{4}x^{-\frac{3}{4}}y^{\frac{1}{4}}$, $u_y = \frac{1}{4}x^{\frac{1}{4}}y^{-\frac{3}{4}}$

解答　(x, y) で極値を取るとすると

$$\begin{cases} \frac{1}{4}x^{-\frac{3}{4}}y^{\frac{1}{4}} + \lambda\cdot(-p) = 0 & (1) \leftrightarrow u_x + \lambda g_x = 0 \\ \frac{1}{4}x^{\frac{1}{4}}y^{-\frac{3}{4}} + \lambda\cdot(-q) = 0 & (2) \leftrightarrow u_y + \lambda g_y = 0 \\ I - px - qy = 0 & (3) \leftrightarrow g = 0 \end{cases}$$

を満たす $\lambda \in \mathbf{R}$ が存在します．(1) において $\lambda = 0$ とすると

$$\frac{1}{4}x^{-\frac{3}{4}}y^{\frac{1}{4}} = 0$$

となりますが,$x, y > 0$ に反します．よって $\lambda \neq 0$ であることが分かります．(1) $\times x$, (2) $\times y$ から

$$x^{\frac{1}{4}}y^{\frac{1}{4}} = 4\lambda px$$
$$x^{\frac{1}{4}}y^{\frac{1}{4}} = 4\lambda qy$$

が従いますが，$\lambda \neq 0$ から $px = qy$ であることが分かります．これから (3) を用いて

$$px = qy = \frac{I}{2} \quad 従って \quad x = \frac{I}{2p}, \quad y = \frac{I}{2q}$$

需要関数

であることが分かります[*1]．さらに (1) から

$$\lambda = \frac{1}{4p}x^{-\frac{3}{4}}y^{\frac{1}{4}} = \frac{1}{2\sqrt{2}}\cdot\frac{1}{I^{\frac{1}{2}}p^{\frac{1}{4}}q^{\frac{1}{4}}}$$

であることも分かります．

補足　2 階の偏微分係数を用いて極大・極小を判定します．

$$g_{*\#} = 0 \quad (*, \# = x, y)$$
$$f_{xx} = -\frac{3}{16}x^{-\frac{7}{4}}y^{\frac{1}{4}},\ f_{xy} = f_{yx} = \frac{1}{16}x^{-\frac{3}{4}}y^{-\frac{3}{4}},\ f_{yy} = -\frac{3}{16}x^{\frac{1}{4}}y^{-\frac{7}{16}}$$

から $L = f + \lambda g$ とおくと

$$B = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix} = \begin{vmatrix} 0 & -p & -q \\ -p & -\frac{3}{16}x^{-\frac{7}{4}}y^{\frac{1}{4}} & \frac{1}{16}x^{-\frac{3}{4}}y^{-\frac{3}{4}} \\ -q & \frac{1}{16}x^{-\frac{3}{4}}y^{-\frac{3}{4}} & -\frac{3}{16}x^{\frac{1}{4}}y^{-\frac{7}{4}} \end{vmatrix}$$
$$= \frac{3}{16}x^{-\frac{7}{4}}y^{\frac{1}{4}}q^2 + \frac{1}{8}x^{-\frac{3}{4}}y^{-\frac{3}{4}}pq + \frac{3}{16}x^{\frac{1}{4}}y^{-\frac{7}{4}}p^2 > 0$$

[*1] この関数 $x = x(p,q,I)$, $y = y(p,q,I)$ を需要関数と呼びます．

から，停留点

$$(x, y, \lambda) = \left(\frac{I}{2p}, \frac{I}{2q}, \frac{1}{2\sqrt{2}} \cdot \frac{1}{I^{\frac{1}{2}} p^{\frac{1}{4}} q^{\frac{1}{4}}}\right)$$

で極大値をとることが分かります．

II 以下の積分の値を求めましょう．
(1) $\int_0^1 t^4 dt$ **(2)** $\int_0^1 e^{2t} dt$ **(3)** $\int_1^{e^2} \frac{1}{t} dt$ **(4)** $\int_0^9 \sqrt{t} dt$
(5) $\int_1^9 \frac{1}{\sqrt{t}} dt$ **(6)** $\int_1^3 \sqrt{t+1} dt$ **(7)** $\int_1^2 \frac{1}{x^2} dx$

(1) $(t^5)' = 5t^4 \Rightarrow \left(\frac{1}{5}t^5\right)' = t^4$

$$\begin{aligned} \int_0^1 t^4 dt &= \int_0^1 \left(\frac{1}{5}t^5\right)' dt \\ &= \left[\frac{1}{5}t^5\right]_0^1 = \frac{1}{5} \end{aligned}$$

(2) $(e^{2t})' = 2e^{2t} \Rightarrow \left(\frac{1}{2}e^{2t}\right)' = e^{2t}$

$$\begin{aligned} \int_0^1 e^{2t} dt &= \int_0^1 \left(\frac{1}{2}e^{2t}\right)' dt \\ &= \left[\frac{1}{2}e^{2t}\right]_0^1 = \frac{1}{2}\left(e^2 - 1\right) \end{aligned}$$

(3) $(\log t)' = \frac{1}{t}$

$$\begin{aligned} \int_1^{e^2} \frac{1}{t} dt &= \int_1^{e^2} (\log t)' dt \\ &= [\log t]_1^{e^2} = 2 \end{aligned}$$

(4) $\log e^2 = 2, \ \log 1 = 0$

$$\begin{aligned} \int_0^9 \sqrt{t} dt &= \int_0^9 \left(\frac{2}{3}t\sqrt{t}\right)' dt \\ &= \left[\frac{2}{3}t\sqrt{t}\right]_0^9 = \cancel{6} \ 18 \end{aligned}$$

$(t\sqrt{t})' = \left(t^{\frac{3}{2}}\right)' = \frac{3}{2}t^{\frac{1}{2}}$

$\left(\frac{2}{3}t\sqrt{t}\right)' = \sqrt{t}.$

$= \frac{2}{\cancel{3}} 9 \cdot \cancel{3}$

(5) $(\sqrt{t})' = \frac{1}{2} \cdot \frac{1}{\sqrt{t}}$

$\rightarrow (2\sqrt{t})' = \frac{1}{\sqrt{t}}.$

$$\begin{aligned} \int_1^9 \frac{1}{\sqrt{t}} dt &= \int_1^9 \left(2\sqrt{t}\right)' dt \\ &= \left[2\sqrt{t}\right]_1^9 = 2(3-1) = 4 \end{aligned}$$

(6) $\left((t+1)\sqrt{t+1}\right)' = \left((t+1)^{\frac{3}{2}}\right)'$

$= \frac{3}{2}(t+1) \cdot 1$

$(t+1)'$

$$\begin{aligned} \int_0^3 \sqrt{t+1} dt &= \int_0^3 \left(\frac{2}{3}(t+1)\sqrt{t+1}\right)' dt \\ &= \left[\frac{2}{3}(t+1)\sqrt{t+1}\right]_0^3 = \frac{14}{3} \end{aligned}$$

$= \frac{2}{3}(4 \cdot 2 - 1)$

(7)

$$\begin{aligned} \int_1^2 \frac{1}{x^2} dx &= \int_1^2 \left(-\frac{1}{x}\right)' dx \\ &= \left[-\frac{1}{x}\right]_1^2 = \frac{1}{2} \end{aligned}$$

$= -\left(\frac{1}{2} - 1\right)$

$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

$\rightsquigarrow \left(-\frac{1}{x}\right)' = \frac{1}{x^2}$

$(\quad)' = \sqrt{2t+1}$

$\left\{(2t+1)^{\frac{3}{2}}\right\}' = \frac{3}{2}(2t+1)^{\frac{1}{2}} \cdot 2$

$\left(\frac{1}{3}(2t+1)^{\frac{3}{2}}\right)' = \sqrt{2t+1}$ $(2t+1)' = 2$

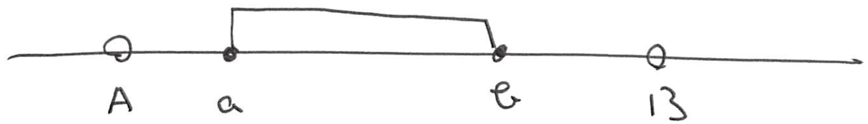
部分積分

t を消すために。(log t を消すため)

$$\int_0^1 t e^t \, dt = \cdots$$

$f : (A, B) \longrightarrow \mathbb{R}$

$g : (A, B) \longrightarrow \mathbb{R}$



$F' = f,\ G' = g$ とする.

Leibnizの公式.

$$(FG)' = F'G + FG' = fG + Fg.$$

不定積分の定義

$$\boxed{\left(\int f(t)\,dt\right)' = f(t)}$$

$$\underset{\substack{\shortparallel \\ (\int fG)'}}{(fG)} = (FG)' - \underbrace{Fg}_{\substack{\shortparallel \\ (\int Fg)'}} = \left(FG - \int Fg\right)'$$

$\leadsto$ ★$_1$ $\displaystyle\int fG\,dt = FG - \int Fg\,dt$

★$_2$ $\displaystyle\int_a^b fG\,dt = [FG]_a^b - \int_a^b Fg\,dt.$

$$\left.\begin{array}{l} F \to f \\ G \to g \end{array}\right\} \Rightarrow \begin{array}{l} f \to f' \\ g \to g' \end{array}$$

$\star_3$ $$\int_a^b f'g \, dt = [fg]_a^b - \int_a^b f g' \, dt$$

131

① $\displaystyle\int_0^1 \overset{g}{t}\,\overset{f'}{e^t}\,dt = \int_0^1 t\,(e^t)'\,dt$ $\qquad (e^t)' = e^t$

$\displaystyle = \left[t e^t\right]_0^1 - \int_0^1 e^t \cdot \underset{(t)'}{1}\,dt$ $\qquad (e^t)' = e^t$

$\displaystyle = e - \left[e^t\right]_0^1$

$\displaystyle = e - (e-1) = 1$

② $\displaystyle\int_1^2 1\cdot\log t\,dt = \int_1^2 (t)'\log t\,dt = \left[t\log t\right]_1^2 - \int_1^2 t\cdot\underset{(\log t)'}{\frac{1}{t}}\,dt$

$\displaystyle = 2\log 2 - \int_1^2 dt$ $\qquad \boxed{\log 1 = 0}$

$\displaystyle = 2\log 2 - 1$ $\qquad \boxed{\log e = 1}$

③ $\displaystyle\int_1^e \overset{f'}{t^2}\,\overset{g}{\log t}\,dt = \int_1^e \left(\frac{t^3}{3}\right)'\log t\,dt = \left[\frac{t^3}{3}\log t\right]_1^e - \int_1^e \frac{t^3}{3}\cdot\overset{(\log t)'}{\frac{1}{t}}\,dt$

$\displaystyle = \frac{e^3}{3} - \frac{1}{3}\int_1^e t^2\,dt$

$\displaystyle = \frac{e^3}{3} - \frac{1}{3}\left[\frac{1}{3}t^3\right]_1^e = \frac{e^3}{3} - \frac{1}{9}(e^3-1) = \frac{2}{9}e^3 + \frac{1}{9}$

$$(e^t)' = e^t$$

(4) $$\int_0^1 \overset{g}{t^2}\,\overset{f'}{e^t}\,dt = \int_0^1 t^2 (e^t)'\,dt = \left[t^2 e^t\right]_0^1 - \int_0^1 \overset{(t^2)'}{2t}\, e^t\,dt$$

$$= e - 2\underbrace{\int_0^1 t\, e^t\,dt}_{=1} = e - 2\cdot 1$$

$$\int_0^1 \underset{g}{t}\,\underset{f'}{e^{2t}}\,dt = \int_0^1 t\left(\frac{1}{2}e^{2t}\right)'dt = \left[t\cdot\frac{1}{2}e^{2t}\right]_0^1 - \int_0^1 \frac{1}{2}e^{2t}\,dt$$

$$= \cdots$$

$$(e^{2t})' = 2e^{2t} \rightsquigarrow \left(\frac{1}{2}e^{2t}\right)' = e^{2t}$$

置換積分.(その1)　　$F'(x) = f(x)$

$$\frac{d}{dt} F(\varphi(t)) = F'(\varphi(t))\,\varphi'(t) = f(\varphi(t))\,\varphi'(t)$$

$$F(x) = \frac{x^3}{3},\quad \underset{\displaystyle F'(x)}{\overset{}{}} f(x) = x^2,\quad \underset{\displaystyle x}{\varphi(t)} = 1+t^2 \rightsquigarrow \varphi'(t) = 2t.$$

$$\left(\frac{(1+t^2)^3}{3}\right)' = (1+t^2)^2 \cdot 2t \rightsquigarrow$$

$$\{\quad\}' = (1+t^2)^2 \cdot t$$

$$\{(1+t^2)^3\}' = 3(1+t^2)\cdot 2t$$

$$\left\{\frac{1}{6}(1+t^2)^3\right\}' = (1+t^2)t$$

$$\int_a^b f(\varphi(t))\,\varphi'(t)\,dt = \Big[F(\varphi(t))\Big]_a^b$$

$$= F(\varphi(b)) - F(\varphi(a)) \qquad \varphi(b) = B,\ \varphi(a) = A.$$

$$= F(B) - F(A)$$

$$= \int_A^B f(x)\,dx$$

公式　$\displaystyle \int_a^b f(\varphi(t))\,\varphi'(t)\,dt = \int_A^B f(x)\,dx.$　　但し　$A = \varphi(a)$

$B = \varphi(b).$

$\longrightarrow$

① $\displaystyle\int_0^1 (1+t^2)^4\, t\, dt$

$x = 1+t^2 \rightsquigarrow dx = (1+t^2)'\,dt$

$(1+t^2)' = 2t$

$\frac{1}{2}(1+t^2)' = t$

$$= \frac{1}{2}\int_0^1 (1+t^2)^4 \underbrace{(1+t^2)'\,dt}_{= dx}$$

対応

t	$0 \nearrow 1$
x	$1 \nearrow 2$

$x = 1+t^2$

$$= \frac{1}{2}\int_1^2 x^4\,dx$$

$$= \frac{1}{2}\left[\frac{1}{5}x^5\right]_1^2 = \frac{1}{10}\cdot 31.$$

$(\log x)' = \frac{1}{x}$

② $$\int_0^1 \frac{t}{1+t^2}\,dt = \frac{1}{2}\int_0^1 \frac{(1+t^2)'}{1+t^2}\,dt = \frac{1}{2}\int_1^2 \frac{1}{x}\,d\cancel{t}\,x = \frac{1}{2}\log 2.$$

$$= \frac{1}{2}\left[\log x\right]_1^2$$

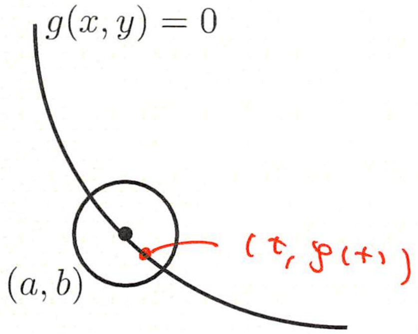
$(\log(1+t^2))' = \dfrac{2t}{1+t^2}$

$\left(\frac{1}{2}\log(1+t^2)\right)' = \dfrac{t}{1+t^2}$

$((1+t^2)^5)' = 5(1+t^2)^4\cdot 2t$

$\left(\frac{1}{10}(1+t^2)^5\right)' = (1+t^2)^4\cdot t$

極大・極小の十分条件

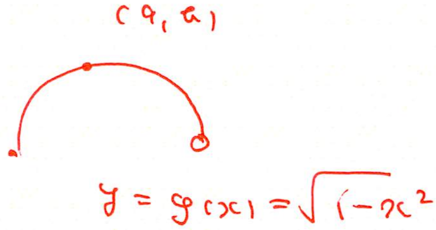


$$g(a,b)=0, \quad g_y(a,b)\neq 0$$

を仮定して, 陰関数定理を適用する. (a,b) の近くで

$$y=\varphi(x)$$

と曲線 $g(x,y)=0$ を表す.



(a,b) で極大（極小）ならば

$$F(t)=f(t,\varphi(t))$$

とすると $F'(a)=0$ が従う.

$$F''(a)>0 \quad (\text{resp.} \quad F''(a)<0$$

ならば (a,b) で極小（resp. 極大）となります.

解法 (2)

Chain Rule を使うと

$$F'(t)=f_x(t,\varphi(t))\cdot 1+f_y(t,\varphi(t))\cdot\varphi'(t)$$

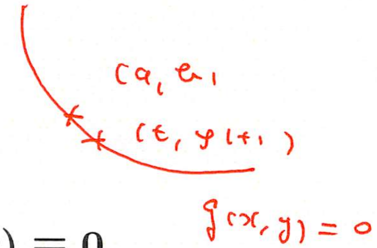
$$\begin{aligned}
F''(t)&=f_{xx}(t,\varphi(t))\cdot 1+f_{xy}(t,\varphi(t))\cdot\varphi'(t)\\
&\qquad+\varphi'(t)\big(f_{yx}(t,\varphi(t))\cdot 1+f_{yy}(t,\varphi(t))\cdot\varphi'(t)\big)\\
&\qquad\quad+f_y(t,\varphi(t))\cdot\varphi''(t)\\
&=f_{xx}(t,\varphi(t))+2f_{xy}(t,\varphi(t))\cdot\varphi'(t)+f_{yy}(t,\varphi(t))\cdot\varphi'(t)^2\\
&\qquad\quad+f_y(t,\varphi(t))\cdot\varphi''(t)
\end{aligned}$$

解法 (3)

$F(t) = f(x(t), y(t))$

$\rightsquigarrow F'(t) = f_x(x(t), y(t)) \cdot x'(t)$

Chain Rule $\quad + f_y(\quad) \; y'(t)$



さらに $g(t, \varphi(t)) \equiv 0$ の両辺を t で微分して

$$g_x(t, \varphi(t)) \cdot 1 + g_y(t, \varphi(t)) \cdot \varphi'(t) \equiv 0$$

$$g_{xx}(t, \varphi(t)) + 2g_{xy}(t, \varphi(t)) \cdot \varphi'(t) + g_{yy}(t, \varphi(t)) \cdot \varphi'(t)^2$$
$$+ g_y(t, \varphi(t)) \cdot \varphi''(t) \equiv 0$$

を得ます.

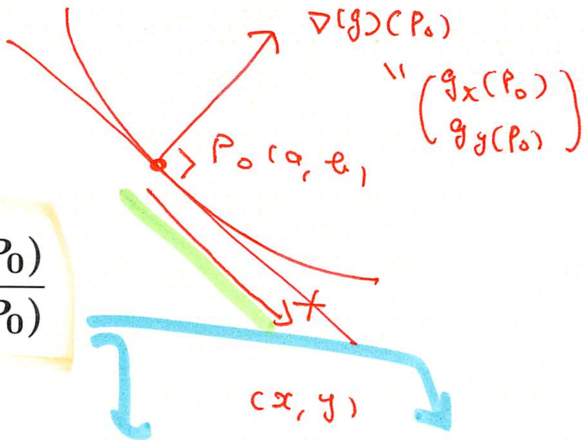
by Young.

$$g_{xx}(\quad) \cdot 1 + g_{xy}(\quad) \cdot \varphi'(t) + \left(g_{yx}(\quad) \cdot 1 + g_{yy}(\quad)\, \varphi'(t)\right) \varphi'(t)$$
$$+ \; g_y(t, \varphi(t)) \cdot \varphi''(t) \equiv 0.$$

解法 (4)

$g_x(a, b)(x - a) + g_y(a, b)(y - b) = 0$

$\iff \left(\nabla(g)(P_0), \begin{pmatrix} x - a \\ y - b \end{pmatrix} \right) = 0$



$t = a$ とするとき $P_0(a, b)$ と定めて

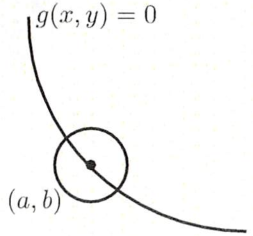
$y = -\dfrac{g_x(P_0)}{g_y(P_0)}(x - a) + b$

$$\varphi'(a) = -\frac{g_x(P_0)}{g_y(P_0)}$$

$$\varphi''(a) = -\frac{1}{g_y(a, b)} \left(g_{xx}(P_0) + 2g_{xy}(P_0) \cdot \varphi'(a) + g_{yy}(P_0) \cdot \varphi'(a)^2 \right)$$

となります.

極大・極小の十分条件



$$g(a,b)=0, \quad g_y(a,b)\neq 0$$

を仮定して, 陰関数定理を適用する. (a,b) の近くで

$$y=\varphi(x)$$

と曲線 $g(x,y)=0$ を表す.

(a,b) で極大（極小）ならば

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とすると $F'(a)=0$ が従う.

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解法 (2)

Chain Rule を使うと

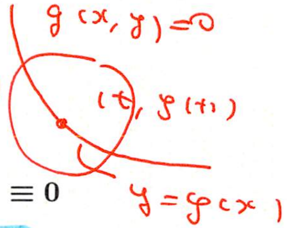
$$F'(t)=f_x(t,\varphi(t))\cdot 1+f_y(t,\varphi(t))\cdot\varphi'(t)$$

$$\begin{aligned}F''(t)&=f_{xx}(t,\varphi(t))\cdot 1+f_{xy}(t,\varphi(t))\cdot\varphi'(t)\\&\quad+\varphi'(t)\left(f_{yx}(t,\varphi(t))\cdot 1+f_{yy}(t,\varphi(t))\cdot\varphi'(t)\right)\\&\quad+f_y(t,\varphi(t))\cdot\varphi''(t)\\&=f_{xx}(t,\varphi(t))+2f_{xy}(t,\varphi(t))\cdot\varphi'(t)+f_{yy}(t,\varphi(t))\cdot\varphi'(t)^2\\&\quad+f_y(t,\varphi(t))\cdot\varphi''(t)\end{aligned}$$

解法 (3)

$F(t)=f(x(t),y(t))$

Chain Rule $\quad F'(t)=f_x(\ )x'(t)+f_y(\ )\ y'(t)$



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$$g_x(t,\varphi(t))\cdot 1+g_y(t,\varphi(t))\cdot\varphi'(t)\equiv 0$$

$$g_{xx}(t,\varphi(t))+2g_{xy}(t,\varphi(t))\cdot\varphi'(t)+g_{yy}(t,\varphi(t))\cdot\varphi'(t)^2$$
$$+\ g_y(t,\varphi(t))\cdot\varphi''(t)\equiv 0$$

を得ます.

$$g_{xx}(\ )\cdot 1+g_{xy}(\ )\cdot\varphi'(t)+\varphi'(t)\left(g_{yx}(\ )\cdot 1+g_{yy}(\ )\cdot\varphi'(t)\right)$$
$$+\ g_y(\ )\cdot\varphi''(t)\equiv 0$$

解法 (4)

$$g_x(a,b)+\underbrace{g_y(a,b)}_{\neq 0}\cdot\varphi'(a)$$

$t=a$ とするとき $P_0(a,b)$ と定めて

$$\varphi'(a)=-\frac{g_x(P_0)}{g_y(P_0)}$$

$$\varphi''(a)=-\frac{1}{g_y(a,b)}\left(g_{xx}(P_0)+2g_{xy}(P_0)\cdot\varphi'(a)+g_{yy}(P_0)\cdot\varphi'(a)^2\right)$$

となります.