2019/07/10
CalcN 2019 L13

I

$$(\tan\theta)' = \left(\frac{\sin\theta}{\cos\theta}\right)'$$

$$= \frac{(\sin\theta)'\cos\theta - \sin\theta\cdot(\cos\theta)'}{\cos^2\theta}$$

$$= \frac{\cos\theta\cdot\cos\theta - \sin\theta\left(-\overset{\sin\theta}{\cancel{\cos\theta}}\right)}{\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta}$$

II (1) $f(t) = \log(1+t)$ のとき　　$u = 1+t,\ \frac{du}{dt} = 1$

$$f'(t) = \frac{1}{1+t}\cdot 1,\quad f''(t) = -\frac{1}{(1+t)^2}\cdot 1,\quad f^{(3)}(t) = \frac{2}{(1+t)^3}\cdot 1$$

$$f^{(4)}(t) = -\frac{3!}{(1+t)^4}$$

で

$$f(0) = \log 1 = 0,\quad f'(0) = 1,\quad f''(0) = -1,$$

$$f^{(3)}(0) = 2$$

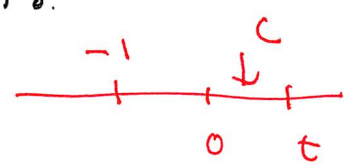
となります。これから

$$\log(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3!}\cdot 2t^3 \overbrace{- \frac{1}{4!}\cdot\frac{3!}{(1+c)^4}}^{f^{(4)}(c)} t^4$$

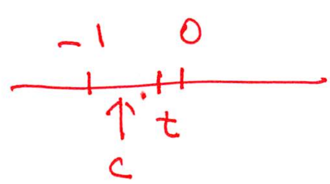
$$= t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}\cdot\frac{1}{(1+c)^4}t^4$$

を満たす c が 0 と t の間に存在する。

真数条件.
$1+t>0$



(2) $t>0$ のとき　$0<c<t$
$-1<t<0$ のとき　$-1<t<c<0$
から　$1+c>0$ が従う。~~まって~~



$t \neq 0$ から $t^4 > 0$ なので

$$-\frac{1}{4}\cdot\frac{1}{(1+c)^4}t^4 < 0$$

が分かる. 従って

$$\log(1+t) < t - \frac{1}{2}t^2 + \frac{1}{3}t^3$$

であることが分かる. $\log(1+t) - \left(t - \frac{1}{2}t^2 + \frac{1}{3}t^3\right) = -\frac{1}{4}\cdot\frac{t^4}{(1+c)^4} < 0$

III $f(t) = \dfrac{1}{1-t}$ のとき $(t \neq 1)$ $u = 1-t$ $\rightarrow \dfrac{du}{dt} = -1$.

$$f'(t) = -\frac{1}{(1-t)^2}\cdot(-1) = \frac{1}{(1-t)^2}$$

$$f''(t) = -\frac{2}{(1-t)^3}\cdot(-1) = \frac{2}{(1-t)^3}$$

$$f^{(3)}(t) = -\frac{2\cdot 3}{(1-t)^4}(-1) = \frac{3!}{(1-t)^4}$$

となる. $f(0)=1$, $f'(0)=1$, $f''(0)=2$ を使う. これから

$$\frac{1}{1-t} = 1 + t + \frac{1}{2}\cdot 2t^2 + \frac{1}{3!}\cdot\frac{3!}{(1-c)^4}t^3$$

$$= 1 + t + t^2 + \frac{1}{(1-c)^4}t^3$$

を満たす c が 0 と t の間に存在する.

Taylor展開

$$f(t)=e^t,\ f'(t)=e^t,\ f''(t)=e^t,\ \cdots,\ f^{(k)}(t)=e^t$$

$$f(0)=1,\ f'(0)=1,\ \cdots$$

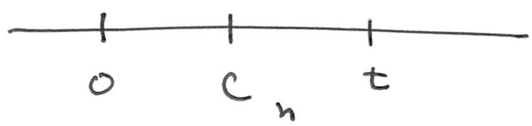
$n+1$ 階の Taylor

$$f(0)+f'(0)t+\frac{1}{2!}f''(0)t^2+\frac{1}{3!}f^{(3)}(0)t^3+\cdots+\frac{f^{(n)}(0)}{n!}t^n+\frac{1}{k!}t^k+$$

$$e^t=1+t+\frac{1}{2!}t^2+\frac{1}{3!}t^3+\cdots+\frac{1}{n!}t^n+\frac{1}{(n+1)!}e^{c_n}t^{n+1}$$

$$+\frac{1}{(n+1)!}f^{(n+1)}(c)\,t^{n+1}$$

をみたす c_n が 0 と t の間に存在

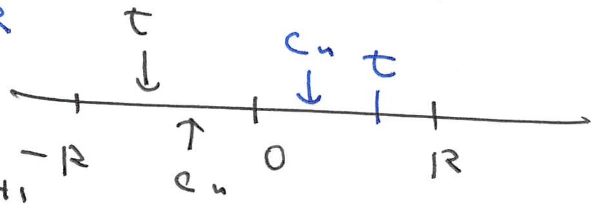


$$e^t-\left(1+t+\frac{1}{2!}t^2+\cdots+\frac{1}{n!}t^n\right)=\frac{1}{(n+1)!}e^{c_n}t^{n+1}$$

$R>0$ として

$|t|\le R$ とすると $|c_n|\le R$

$c_n\le R \rightsquigarrow e^{c_n}\le e^R$



$$0\le\left|\qquad\right|\le\frac{1}{(n+1)!}e^R R^{n+1}$$

$$\downarrow\qquad\qquad\downarrow$$

$$0\qquad\qquad 0$$

$R>0$ を止めて.

$$\boxed{\frac{1}{n!}R^n\longrightarrow 0}$$

コア T にある.

$$\rightsquigarrow\ e^t=\sum_{k=0}^{+\infty}\frac{1}{k!}t^k$$

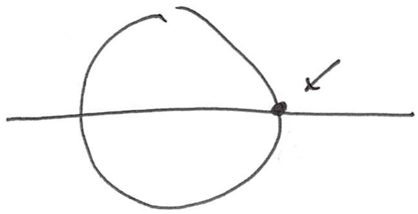
e^t の $t=0$ における Taylor 展開.

$0!=1$ という約束

マクローリン展開

$$=1+t+\frac{1}{2!}t^2+\frac{1}{3!}t^3+\cdots$$

$$f(\theta) = \sin\theta. \qquad f(0) = 0$$



$$f'(\theta) = \cos\theta \qquad f'(0) = 1$$

$$f''(\theta) = -\sin\theta \qquad f''(0) = 0$$

$$f^{(3)}(\theta) = -\cos\theta \qquad f^{(3)}(0) = -1$$

$$f^{(4)}(\theta) = \sin\theta \qquad f^{(4)}(0) = 0$$

$f^{(k)}(\theta) = \pm\sin\theta,\ \pm\cos\theta$ のどれか

n階のTaylorの定理

$\theta \neq 0$ のとき

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7 + - \cdots + \frac{1}{(n-1)!} f^{(n-1)}(0)\,\theta^{n-1} + \frac{1}{n!} f^{(n)}(c)\,\theta^n$$

を満たす c が 0 と θ の間に存在.

$$\left| f^{(k)}(\theta) \right| \leq 1 \qquad |\theta| \leq R \text{ とする}$$

$$\left| \frac{1}{n!} f^{(n)}(c)\,\theta^n \right| \leq \frac{1}{n!} R^n \longrightarrow 0$$

$$\leadsto \quad \sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7 + - \cdots$$

$$\cos\theta = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \frac{1}{8!}\theta^8 - + \cdots$$

$$0 \leq \left| \sin\theta - \left(\theta - \frac{1}{3!}\theta^3 + - \cdots + \frac{1}{(n-1)!} f^{(n-1)}(0)\, \theta^{n-1}\right) \right|$$

$$= \frac{1}{n!} \left| f^{(n)}(c)\, \theta^n \right| \qquad |\theta| \leq R$$

$$\leq \frac{1}{n!} R^n$$

$$\downarrow$$

$$0$$

$$\boxed{\frac{R^n}{n!} \longrightarrow 0 \quad (n \to +\infty)}$$

$\rightsquigarrow$

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7 + - \cdots.$$

$\sin\theta$ のマクローリン展開

$$\cos\theta = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + - \cdots.$$

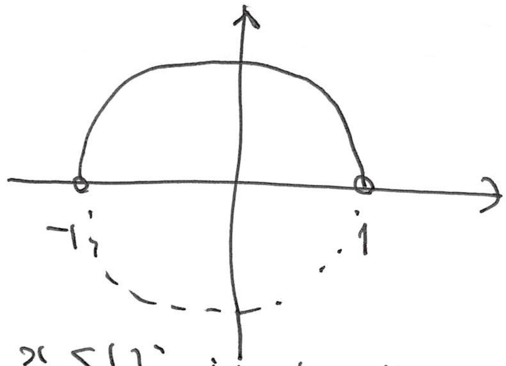
$$\log(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + - \cdots$$

$$|t| < 1$$

$$y = \varphi(x) = \sqrt{1-x^2} \qquad -1 < x < 1$$

$\vee$ 0

$$y^2 = 1 - x^2$$



$$x^2 + y^2 = 1$$

$\rightarrow \quad x^2 + \varphi(x)^2 = 1$ $-1 < x < 1$ で identically となる　両辺を (x で) 微分.

$$2x + 2\varphi(x)\varphi'(x) = 0$$

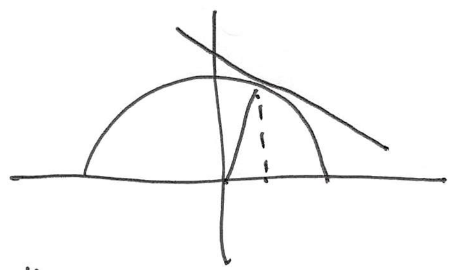
$$\varphi'(x) = -\frac{x}{\varphi(x)}$$

$$u = \varphi(x)$$

$$\frac{du^2}{dx} = \frac{du^2}{du} \cdot \frac{du}{dx} = 2u \cdot \frac{du}{dx} = 2\varphi(x) \cdot \varphi'(x)$$

$$x = \frac{1}{2}. \quad \varphi\left(\frac{1}{2}\right) = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$$

$$\varphi'\left(\frac{1}{2}\right) = -\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$



x で微分

$$2x + 2\varphi(x)\varphi'(x) = 0$$

$$2 + 2\varphi'(x)^2 + 2\varphi(x)\varphi''(x) \equiv 0$$

$$\varphi''(x) = -\frac{1 + \varphi'(x)^2}{\varphi(x)} = -\frac{1 + \frac{x^2}{\varphi(x)^2}}{\varphi(x)}$$

$$= -\frac{\varphi(x)^2 + x^2}{\varphi(x)^3} = -\frac{1}{\varphi(x)^3}$$

2項分布 $\quad p, q > 0 \quad p + q = 1$

$$X = 0, 1, 2, \cdots, n$$

$$P(X = k) = {}_nC_k\, p^k q^{n-k}.$$

$$E[X] = \sum_{k=0}^{n} k\, {}_nC_k\, p^k q^{n-k}$$

1

$$\boxed{k\, {}_nC_k = n\, {}_{n-1}C_{k-1}}$$

$$(p+q)^n = \sum_{k=0}^{n} {}_nC_k\, p^k q^{n-k}$$

$$= q^n + {}_nC_1\, q^{n-1} p + {}_nC_2\, q^{n-2} p^2 + \cdots$$

$$\cdots + {}_nC_k\, q^{n-k} p^k + \cdots + p^n$$

pで微分する

$$n(p+q)^{n-1} = 0 + {}_nC_1\, q^{n-1} \cdot 1 + {}_nC_2\, q^{n-2} \cdot 2p + \cdots$$

$$\cdots + {}_nC_k\, q^{n-k}\, k p^{k-1} + \cdots + n p^{n-1}$$

もう1回 pで微分

$$np(p+q)^{n-1} = {}_nC_1\, q^{n-1} \cdot p + {}_nC_2\, q^{n-2} \cdot 2p^2 + \cdots$$

$1 \cdot p$

$$\cdots + {}_nC_k\, q^{n-k}\, k p^k + \cdots + n p^n$$

$$E[X] = np(p+q)^{n-1} \underset{p+q=1}{=} np.$$

2次のモーメント

$$E[X^2] = \sum_{k=0}^{n} \overset{k(k-1)+k}{\overset{\shortparallel}{k^2}} {}_nC_k \, q^{n-k} p^k$$

$$= \sum_{k=0 \to 2}^{n} k(k-1) \, {}_nC_k \, q^{n-k} p^k$$

$$+ \underbrace{\sum_{k=0}^{n} k \, {}_nC_k \, q^{n-k} p^k}_{\shortparallel \atop np.}$$

$= \cdots$

I

(1) $f(\theta) = \cos\theta$ に対して 3階の Taylor の定理を適用する.

(2) $\cos\theta \;\boxed{}\; 1 - \frac{1}{2}\theta^2 \quad \left(0 < \theta < \frac{\pi}{2}\right)$

$\cos\theta \;\boxed{}\; 1 - \frac{1}{2}\theta^2 \quad \left(-\frac{\pi}{2} < \theta < 0\right)$

II $\quad y = \varphi(x) = \sqrt{9 - \frac{x^2}{4}}$

が $\quad \frac{x^2}{4} + \varphi(x)^2 = 9$

を満たすことを用いて

$\varphi'(x)$, $\varphi''(x)$ を x, $\varphi(x)$ で書く.

I
(1) $f(\theta)=\sin\theta$ に 4階の Taylor の定理を適用.

(2)

$$\sin\theta \ \boxed{} \ \theta-\frac{1}{3!}\theta^3 \quad \left(0<\theta<\frac{\pi}{2}\right)$$

$$\sin\theta \ \boxed{} \ \theta-\frac{1}{3!}\theta^3 \quad \left(-\frac{\pi}{2}<\theta<0\right)$$

II

$$y=\varphi(x)=\sqrt{4-\frac{x^2}{9}}$$

が $\dfrac{x^2}{9}+\varphi(x)^2-\underset{4}{\cancel{9}}\equiv 0$ φ φ

を満たすことを用いて

$\varphi'(x)$, $\varphi''(x)$ を x, $\varphi(x)$ で表す.