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I (1) $\int_0^{\frac{\pi}{2}} t\sin t dt$, **(2)** $\int_{-1}^{1} \frac{1}{\sqrt{x+2}} dx$, **(3)** $\int_0^1 x(x-1)^3 dx$, **(4)** $\int_0^6 \left(\frac{1}{3}x-1\right)^4 dx$, **(5)** $\int_{-3}^{-1} \frac{1}{(2x+1)^3} dx$, **(6)** $\int_0^1 (x+1)e^x dx$, **(7)** $\int_{-1}^{1} (x+1)^3(x-1)dx$ (by integration by parts)

Solution (1) Since $(\cos t)' = -\sin t$, we find $(-\cos t)' = \sin t$.

$$\begin{aligned}\int_0^{\frac{\pi}{2}} t\sin t dt &= \int_0^{\frac{\pi}{2}} t(-\cos t)' dt = -\left[t\cos t\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos t dt \\ &= -\left(\frac{\pi}{2}\cdot\cos\frac{\pi}{2} - 0\cdot\cos 0\right) + \left[\sin t\right]_0^{\frac{\pi}{2}} \\ &= \sin\frac{\pi}{2} - \sin 0 = 1\end{aligned}$$

(2) Since $\left(\sqrt{x+2}\right)' = \frac{1}{2}\cdot\frac{1}{\sqrt{x+2}}$, we find $\left(2\sqrt{x+2}\right)' = \frac{1}{\sqrt{x+2}}$.

$$\int_{-1}^{1} \frac{1}{\sqrt{x+2}} dx = \left[2\sqrt{x+2}\right]_{-1}^{1} = 2\left(\sqrt{3}-1\right)$$

(3)

$$\begin{aligned}\int_0^1 x(x-1)^3 dx &= \int_0^1 x\left\{\frac{(x-1)^4}{4}\right\}' dx \\ &= \left\{x\cdot\frac{(x-1)^4}{4}\right\}_0^1 - \int_0^1 \frac{(x-1)^4}{4} dx \\ &= -\frac{1}{4}\left[\frac{(x-1)^5}{5}\right]_0^1 = \frac{1}{20}\end{aligned}$$

Another Solution

$$\begin{aligned}\int_0^1 x(x-1)^3 dx &= \int_0^1 (x-1+1)(x-1)^3 dx \\ &= \int_0^1 (x-1)^4 dx + \int_0^1 (x-1)^3 dx \\ &= \left[\frac{(x-1)^5}{5}\right]_0^1 + \left[\frac{(x-1)^4}{4}\right]_0^1 \\ &= -\frac{(-1)^5}{5} + \frac{(-1)^4}{4} = \frac{1}{20}\end{aligned}$$

(4) It follows from $\left\{\left(\frac{x}{3}-1\right)^5\right\}' = \frac{5}{3}\left(\frac{x}{3}-1\right)^4$ that

$$\frac{3}{5}\left\{\left(\frac{x}{3}-1\right)^5\right\}' = \left(\frac{x}{3}-1\right)^4$$

The we get

$$\int_0^6 \left(\frac{1}{3}x-1\right)^4 dx = \left[\frac{3}{5}\left(\frac{x}{3}-1\right)^5\right]_0^6 = \frac{3}{5}\left(1^5-(-1)^5\right) = \frac{6}{5}$$

(5) Since $\left\{\frac{1}{(2x+1)^2}\right\}' = -\frac{2\cdot 2}{(2x+1)^3} = -\frac{4}{(2x+1)^3}$ we find $\left\{-\frac{1}{4}\cdot\frac{1}{(2x+1)^2}\right\}' = \frac{1}{(2x+1)^3}$

$$\begin{aligned}\int_{-3}^{-1}\frac{1}{(2x+1)^3}dx &= \left\{-\frac{1}{4}\cdot\frac{2\cdot 2}{(2x+1)^2}\right\}_{-3}^{-1}\\ &= -\frac{1}{4}\left(1-\frac{1}{25}\right) = -\frac{6}{25}\end{aligned}$$

(6)

$$\begin{aligned}\int_0^1 (x+1)e^x dx &= \int_0^1 (x+1)(e^x)'dx\\ &= [(x+1)e^x]_0^1 - \int_0^1 e^x dx\\ &= 2e-1-[e^x]_0^1 = 2e-1-(e-1) = e\end{aligned}$$

(7)

$$\begin{aligned}\int_{-1}^1 (x+1)^3(x-1)dx &= \int_{-1}^1 \left(\frac{(x+1)^4}{4}\right)'(x-1)dx\\ &= \left[\frac{(x+1)^4}{4}\cdot(x-1)\right]_{-1}^1 - \int_{-1}^1\frac{(x+1)^4}{4}dx\\ &= -\frac{1}{4}\left[\frac{(x+1)^5}{5}\right]_0^1 = -\frac{1}{4}\cdot\frac{2^5}{5} = -\frac{8}{5}\end{aligned}$$

II (1) $\int_{-1}^2 \frac{x}{\sqrt{3-x}}dx$, **(2)** $\int_0^1 \frac{x-1}{(2-x)^2}dx$, **(3)** $\int_1^2 x\sqrt{2-x}dx$, **(4)** $\int_0^6 \left(\frac{x}{3}-1\right)^4 dx$, **(5)** $\int_1^2 \frac{e^x}{e^x+1}dx$, **(6)** $\int_1^2 \frac{e^x}{(e^x+1)^2}dx$, **(7)** $\int_1^e \frac{(\log x)^2}{x}dx$, **(8)** $\int_0^1 \sqrt{3-2x}dx$

Solution (1) We make a substitution by $x=\varphi(t) := 3-t$ so that we have $t = 3-x$. Then it follows $\varphi'(t) = -1$, $\varphi(2) = 1$, $\varphi(-1) = 4$ and thus the correspondence of the intervals of integration

$$\begin{array}{c|ccc} t & 4 & \searrow & 1 \\ \hline x & -1 & \nearrow & 2 \end{array}$$

Thus we get

$$\begin{aligned}\int_{-1}^2 \frac{x}{\sqrt{3-x}}dx &= \int_4^1 \frac{3-t}{\sqrt{t}}(-1)dt\\ &= \int_1^4\left(\frac{3}{\sqrt{t}}-\sqrt{t}\right)dt\\ &= 3\left[2\sqrt{t}\right]_1^4 - \left[\frac{2}{3}t\sqrt{t}\right]_1^4\\ &= 3\cdot 2(1-0) - \frac{2}{3}(8-1) = \frac{4}{3}\end{aligned}$$

Another Solution We make a substitution by $x = \varphi(t) := 3 - t^2$ so that we have $t = \sqrt{3-x}$. Since $t \geq 0$, $x = -1$ corresponds to $t = \sqrt{4} = 2$ and $x = 2$ to $t = \sqrt{1} = 1$. Avccordingly the correspondence of the intervals of the integration follows.

$$\begin{array}{c|ccc} t & 2 & \searrow & 1 \\ \hline x & -1 & \nearrow & 2 \end{array}$$

Moreover we have $\varphi'(t) = -2t$ and thus

$$\begin{aligned} \int_{-1}^{2} \frac{x}{\sqrt{3-x}} dx &= \int_{2}^{1} \frac{3-t^2}{t} \cdot (-2t) dt \\ &= 2\int_{1}^{2} (3-t^2) dt \\ &= 6\,[t]_1^2 - 2\left[\frac{t^2}{3}\right]_1^2 \\ &= 6 - 2\frac{8-1}{3} = \frac{4}{3} \end{aligned}$$

(2) We make a substitution by $x = \varphi(t) := 2 - t$ so that $t = 2 - x$ holds. Then the correspondence of the intervals of integration follows:

$$\begin{array}{c|ccc} t & 2 & \searrow & 1 \\ \hline x & 0 & \nearrow & 1 \end{array}$$

Moreover we have $\varphi'(t) = -1$ and thus

$$\begin{aligned} \int_{0}^{1} \frac{x-1}{(2-x)^2} dx &= \int_{2}^{1} \frac{1-t}{t^2}(-1) dt \\ &= \int_{1}^{2} \left(\frac{1}{t^2} - \frac{1}{t}\right) dt \\ &= \left[-\frac{1}{t}\right]_1^2 - [\log t]_1^2 \\ &= \left(-\frac{1}{2} + 1\right) - (\log 2 - \log 1) = \frac{1}{2} - \log 2 \end{aligned}$$

(3) We make a substitution $x = \varphi(t) := 2 - t$ so that $t = 2 - x$ holds.

$$\begin{array}{c|ccc} t & 2 & \searrow & 1 \\ \hline x & 0 & \nearrow & 1 \end{array}$$

Moreover we have $\varphi'(t) = -1$ and thus

$$\begin{aligned} \int_{1}^{2} x\sqrt{2-x} dx &= \int_{1}^{0} (2-t)\sqrt{t}(-1) dt \\ &= \int_{0}^{1} \left(2\sqrt{t} - t\sqrt{t}\right) dt \\ &= 2\left[\frac{2}{3}t\sqrt{t}\right]_0^1 - \left[\frac{2}{5}t^2\sqrt{t}\right]_0^1 \\ &= \frac{4}{3} - \frac{2}{5} = \frac{14}{15} \end{aligned}$$

(4) We make a substitution $y = \varphi(x) = \frac{x}{3} - 1$ to get $\varphi'(x) = \frac{1}{3}$ and the correspondence of the intervals of integration

$$\begin{array}{c|ccc} x & 0 & \nearrow & 6 \\ \hline y & -1 & \nearrow & 1 \end{array}$$

The we get

$$\begin{aligned} \int_0^6 \left(\frac{1}{3}x - 1\right)^4 dx &= 3\int_0^6 \left(\frac{1}{3}x - 1\right)^4 \left(\frac{1}{3}x - 1\right)' dx \\ &= \frac{1}{3}\int_{-1}^1 y^4 dy = 3 \cdot \left[\frac{y^5}{5}\right]_{-1}^1 = 3 \cdot \frac{1^5 - (-1)^5}{5} = \frac{6}{5} \end{aligned}$$

(5) We get a substitution $y = \varphi(x) = e^x + 1$ to get $\varphi'(x) = e^x$, $\varphi(2) = e^2 + 1$, $\varphi(1) = e + 1$. Then it follows

$$\begin{aligned} \int_1^2 \frac{e^x}{e^x + 1} dx &= \int_{e+1}^{e^2+1} \frac{1}{y} dy \\ &= [\log y]_{e+1}^{e^2+1} = \log(e^2 + 1) - \log e + 1 = \log \frac{e^2 + 1}{e + 1} \end{aligned}$$

(6) We make a substitution $y = \varphi(x) = e^x + 1$ to get $\varphi'(x) = e^x$, $\varphi(2) = e^2 + 1$, $\varphi(1) = e + 1$. Then it follows

$$\begin{aligned} \int_1^2 \frac{e^x}{(e^x + 1)^2} dx &= \int_{e+1}^{e^2+1} \frac{1}{y^2} dy \\ &= \left[-\frac{1}{y}\right]_{e+1}^{e^2+1} = -\frac{1}{e+1} + \frac{1}{e^2+1} = \frac{e(e-1)}{(e+1)(e^2+1)} \end{aligned}$$

(7) We make a substitution $x = \varphi(x) = \log t$ to get $\varphi'(t) = \frac{1}{t}$, $\varphi(e) = 1$, $\varphi(1) = 0$. Then it follows

$$\int_1^e \frac{(\log t)^2}{t} dt = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

(8) We make a substitution $y = \varphi(x) = 3 - 2x$ to get $\varphi'(y) = -2$. Moreover we have the correspondences $\varphi(1) = 1$, $\varphi(0) = 3$. Then it follows

$$\begin{aligned} \int_0^1 \sqrt{3 - 2x} dx &= -\frac{1}{2}\int_0^1 \sqrt{3 - 2x}(3 - 2x)' dx \qquad = -\frac{1}{2}\int_3^1 \sqrt{y} dy = \frac{1}{2}\int_1^3 \sqrt{y} dy \\ &= \frac{1}{2}\left[y\sqrt{y}\right]_1^3 = \frac{1}{3}(3\sqrt{3} - 1) \end{aligned}$$