(1) $\int_0^1 t^4\,dt = \left[\frac{1}{5}t^5\right]_{t=0}^{t=1} = \frac{1}{5}$

$(t^5)' = 5t^4 \qquad \left(\frac{1}{5}t^5\right)' = t^4$

(2) $\int_0^1 e^t\,dt = \left[e^t\right]_{t=0}^{t=1} = e-1$

$(e^t)' = e^t$

(3) $\int_1^2 \frac{1}{t^2}\,dt = \left[-\frac{1}{t}\right]_1^2 = -\left(\frac{1}{2}-\frac{1}{1}\right) = \frac{1}{2}$.

$\left(\frac{1}{t}\right)' = -\frac{1}{t^2} \qquad \left(-\frac{1}{t}\right)' = \frac{1}{t^2}$

(4) $\int_1^4 x\sqrt{x}\,dx = \left[\frac{2}{5}x^{\frac{5}{2}}\right]_1^4 = \frac{2}{5}(4\cdot4\cdot2-1) = \frac{62}{5}$.

$\left(x^{\frac{5}{2}}\right)' = \frac{5}{2}x^{\frac{3}{2}} \qquad \left(\frac{2}{5}x^{\frac{5}{2}}\right)' = x^{\frac{3}{2}}$

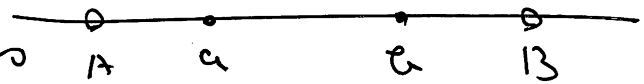
(5) $\int_0^{\frac{\pi}{2}} \cos t\,dt = \left[\sin t\right]_0^{\frac{\pi}{2}} = 1-0 = 1$

(6) $\displaystyle\int_1^2 (x-1)^3\,dx = \left[\frac{1}{4}(x-1)^4\right]_1^2 = \frac{1}{4}$

$\{(x-1)^4\}' = 4(x-1)^3 \qquad \left\{\frac{1}{4}(x-1)^4\right\}' = (x-1)^3$

Integration by parts

$$\left.\begin{aligned} f &: (A, B) \longrightarrow \mathbb{R} \\ g &: (A, B) \longrightarrow \mathbb{R} \end{aligned}\right\} \text{continuous}$$



$$\left.\begin{aligned} F(x) &= \int_a^x f(t)\,dt \\ G(x) &= \int_a^x g(t)\,dt \end{aligned}\right\} \longrightarrow F' = f, \quad G' = g.$$

$$\int_a^b fG = \cdots$$

$$(FG)' = F'G + F \cdot G' = fG + F \cdot g$$

$$fG = (FG)' - Fg.$$

$$\rightsquigarrow \int fG\,dx = FG - \int Fg\,dx$$

$$\rightsquigarrow \int_a^b fG\,dx = [FG]_a^b - \int_a^b Fg\,dx.$$

$F \rightsquigarrow f, \quad G \rightsquigarrow g.$

Rules:

$$\int f' g \, dx = fg - \int f g' \, dx$$

$$\int_a^b f' g \, dx = [fg]_a^b - \int_a^b f g' \, dx$$

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① $\displaystyle \int_0^1 t e^t \, dt = \int_0^1 t (e^t)' \, dt = [t e^t]_0^1 - \int_0^1 \underbrace{1}_{(t)'} e^t \, dt$

$(e^t)' = e^t$

$= 1 \cdot e - 0 \cdot 1 - [e^t]_0^1$

$= e - (e - 1) = 1$

1.

② $\displaystyle \int_1^2 \log t \, dt = \int_1^2 (t)' \log t \, dt.$

$= [t \log t]_1^2 - \int_1^2 t \cdot (\log t)' \, dt$

$= 2 \log 2 - 1 \cdot \underbrace{\log 1}_{= 0} - \int_1^2 dt$

$= 2 \log 2 - 1$

$$\int_1^2 t^n \log t\, dt = \int_1^2 \left(\frac{t^{n+1}}{n+1}\right)' \log t\, dt = \cdots$$

③ $$\int_1^e x \log x\, dx = \int_1^e \left(\frac{x^2}{2}\right)' \log x\, dx$$

$$= \left[\frac{x^2}{2} \cdot \log x\right]_1^e - \int_1^e \frac{x^2}{2} \cdot \underbrace{\frac{1}{x}}_{= (\log x)'} dx$$

$$= \frac{e^2}{2} \cdot \log e - \frac{1}{2} \cdot \underbrace{\log 1}_{=0} - \int_1^e \frac{1}{2} \cdot x\, dx$$

$$= \frac{e^2}{2} - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e = \frac{e^2}{2} - \frac{1}{4}(e^2 - 1) = \frac{e^2+1}{4}$$

Integration by substitution

$$F'(x) = f(x)$$

$$\left(F(\varphi(t)) \right)' = \frac{dF}{dx}(\varphi(t)) \cdot \varphi'(t)$$

$$= f(\varphi(t))\, \varphi'(t)$$

$$\int_a^b f(\varphi(t))\, \varphi'(t)\, dt = \Big[F(\varphi(t)) \Big]_{t=a}^{t=b}$$

$$= F(\varphi(b)) - F(\varphi(a))$$

$B = \varphi(b)$

$A = \varphi(a)$

$$= \Big[F(x) \Big]_A^B$$

$$= \int_A^B f(x)\, dx.$$

$$\boxed{\int_a^b f(\varphi(t))\, \underbrace{\varphi'(t)\, dt}_{= dx} = \int_A^B f(x)\, dx}$$

dx.

① $\int_0^1 (1+t^2)^2 \cdot t\,dt = \frac{1}{2}\int_0^1 (1+t^2)^2 (1+t^2)'\,dt$

$(1+t^2)' = 2t \rightsquigarrow \frac{1}{2}(1+t^2)' = t$

$f(x) = x^2$

$x = 1+t^2 \rightsquigarrow dx = 2t \cdot dt \rightsquigarrow t \cdot dt = \frac{1}{2}dx.$

$x = \varphi(t) = 1+t^2$

$= \boxed{\frac{1}{2}\int_1^2} x^2 \boxed{dx.}$

t	$0 \nearrow 1$
x	$1 \nearrow 2$

$= \frac{1}{2}\left[\frac{x^3}{3}\right]_1^2 = \frac{1}{2}\left(\frac{8}{3} - \frac{1}{3}\right) = \frac{7}{6}.$

② $\int_0^1 \frac{t}{1+t^2}\,dt = \frac{1}{2}\int_0^1 \frac{(1+t^2)'}{1+t^2}\,dt = \frac{1}{2}\int_1^2 \frac{1}{x}\,dx = \frac{1}{2}[\log x]_1^2$

$f(x) = \frac{1}{x}, \quad \varphi(t) = 1+t^2$

$= \frac{1}{2}(\log 2 - \log 1)$

$= \frac{1}{2}\log 2$

③ $\displaystyle\int_{-3}^{-1} \frac{dt}{(2t+1)^3}$

$\varphi(t) = 2t+1$

$\varphi'(t) = 2 \qquad \varphi'(t)\,dt = 2\,dt = dx$

$\rightarrow dt = \frac{1}{2}dx$

$$= \frac{1}{2}\int_{-5}^{-1} \frac{dx}{x^3}$$

t	$-3 \rightarrow -1$
x	$-5 \rightarrow -1$

$$= \frac{1}{2}\left[-\frac{1}{2}\cdot\frac{1}{x^2}\right]_{-5}^{-1}$$

$\left(\frac{1}{x^2}\right)' = -\frac{2}{x^3} \rightarrow \left(-\frac{1}{2}\cdot\frac{1}{x^2}\right)' = \frac{1}{x^3}$

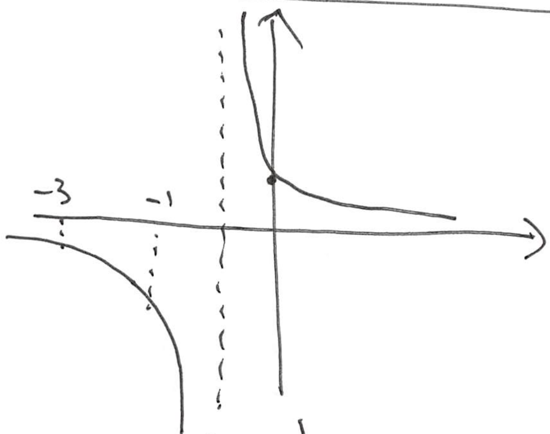
$$= -\frac{1}{4}\left(\frac{1}{1} - \frac{1}{25}\right)$$

$$= -\frac{1}{4}\cdot\frac{24}{25} = -\frac{6}{25}$$

$$\left(\frac{1}{(2t+1)^2}\right)'$$

$$= -\frac{2\cdot 2}{(2t+1)^3}$$

$$\left(-\frac{1}{4}\cdot\frac{1}{(2t+1)^2}\right)' = \frac{1}{(2t+1)^3}$$

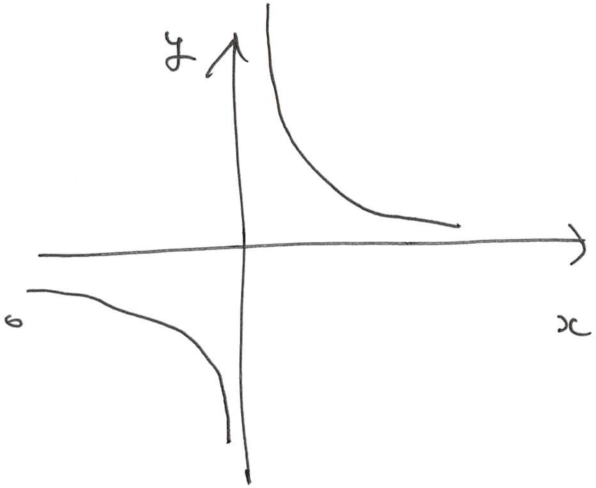


$t = -\frac{1}{2}$.

$$z = f(x, y) = x + 2y$$

subject to the constraint $g(x,y) = 1 - xy = 0$

objective function



$g_x = -y$, $g_y = -x$.

$f_x = 1$, $f_y = 2$

If f is maximal or minimal at (x, y) subject to $g(x,y) = 0$

$\exists \lambda \in \mathbb{R}$

$$\begin{cases} 1 + \lambda(-y) = 0 & (1) \iff f_x + \lambda g_x = 0 \\ 2 + \lambda(-x) = 0 & (2) \iff f_y + \lambda \cdot g_y = 0 \\ 1 - xy = 0 & (3) \iff g = 0 \end{cases}$$

If $\lambda = 0$ in (1), $1 = 0$. It impossible. We assume $\lambda \neq 0$.

$$x = \frac{2}{\lambda}, \quad y = \frac{1}{\lambda}$$

$$1 - \frac{2}{\lambda} \cdot \frac{1}{\lambda} = 0 \iff \lambda^2 = 2 \iff \lambda = \pm\sqrt{2}$$

$$(x, y, \lambda) = \left(\pm\sqrt{2},\ \pm\frac{\sqrt{2}}{2},\ \pm\sqrt{2}\right)$$

Double signs correspond.

複号同順。

$g_{xx} = g_{yy} = 0$, $g_{xy} = g_{yx} = -1$

$f_{xx} = 0$

$L = f + \lambda g$ to find

Bordered Hessian.

$$B = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix} = \begin{vmatrix} 0 & -x & -y \\ -x & 0 & -\lambda \\ -y & -\lambda & 0 \end{vmatrix}$$

$$= -2\lambda x y \quad = -2\lambda$$

At $\left(\sqrt{2}, \frac{\sqrt{2}}{2}, \sqrt{2}\right)$ $\quad B = -2\sqrt{2} < 0 \rightsquigarrow$ Minimal

At $\left(-\sqrt{2}, -\frac{\sqrt{2}}{2}, -\sqrt{2}\right)$ $\quad B = 2\sqrt{2} > 0 \rightsquigarrow$ Maximal

(1) $\int_0^{\pi} \sin t \, dt$

(2) $\int_0^{\frac{\pi}{2}} t \cos t \, dt = \int_0^{\frac{\pi}{2}} t (\sin t)' \, dt$

(3) $\int_1^{e} x^2 \log x \, dx = \int_1^{e} \left(\frac{x^3}{3}\right)' \log x \, dx$

(4) $\int_0^1 t \underbrace{e^{-2t}}_{= (* e^{-2t})'} \, dt$ Hint $(e^{-2t})' = ?$

(5) $\int_0^1 x (x+1)^4 \, dx. = \int_0^1 x \left(\frac{(x+1)^5}{5}\right)' dx$

(6) $\int_0^1 \frac{dt}{3t+2}$ $\varphi(t) = 3t+2$ $\underbrace{\varphi'(t)\, dt}_{} = 3\, dt$, $dx = \varphi'(t)\, dt$

(7) $\int_0^1 \frac{dt}{(3t+2)^2}$

(8) $\int_0^1 \sqrt{1+t^2} \cdot \underbrace{t}_{= \frac{1}{2}(1+t^2)'} \, dt.$