# Chain Rule

Nobuyuki TOSE

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# Chain Rule

- Given an open subset $U$ in $\mathbf{R}^2$ and a function on $U$

$$f: \ U \longrightarrow \mathbf{R}$$

  We assume that $f$ is of $C^2$ class.
- Also given a differentiable curve in $U$

$$(A, B) \longrightarrow U \quad t \mapsto (x(t), y(t))$$

- The we define $F: \ (A, B) \to \mathbf{R}$ by

$$F(t) = f(x(t), y(t))$$

# Chain Rule(2)

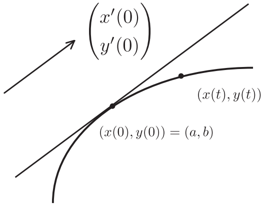
**Theorem**

$$F'(t) = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)$$

# Tangent direction of a curve (1)

The tangent direction of the curve at $\mathrm{P}_0(a,b) = (x(0), y(0))$ is

$$\begin{pmatrix} x'(0) \\ y'(0) \end{pmatrix}$$



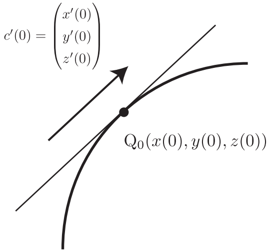
# Tangent direction of a curve (2)

Given a curve in the 3-dimensional space

$$c:\ (A,B)\to \mathbf{R}^3 \quad t\mapsto (x(t),y(t),z(t))$$

Then the tangent vector of $c$ at $\mathrm{Q}_0(x(0),y(0),z(0))$ on $c$ is given by

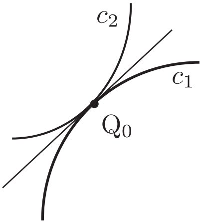
$$c'(0)=\begin{pmatrix}x'(0)\\ y'(0)\\ z'(0)\end{pmatrix}$$



# Two curves tangent to each other

We are given two curves

$$c_1 : (A, B) \to \mathbf{R}^3 \quad t \mapsto (x_1(t), y_1(t), z_1(t))$$
$$c_2 : (A, B) \to \mathbf{R}^3 \quad t \mapsto (x_2(t), y_2(t), z_2(t))$$




We assume that a point $\mathrm{Q}_0(a, b, c)$ is shared by the both curves. Namely

$$(a, b, c) = (x_1(t_1), y_1(t_1), z_1(t_1)) = (x_2(t_2), y_2(t_2), z_2(t_2))$$

holds for some $t_1, t_2 \in (A, B)$. In this situation,

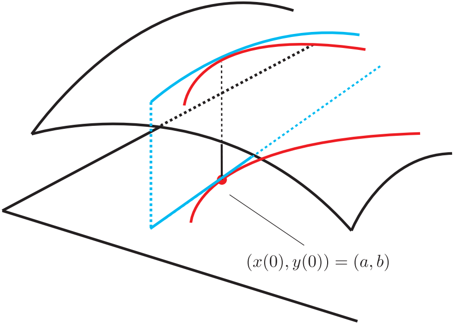
$$c_1 \text{ and } c_2 \text{ are tangent at } \mathrm{Q}_0 \Leftrightarrow C_1'(t_1) \parallel C_2'(t_2)$$

# Two curves tangent to each other

We find a curve in the space

$$(x(t), y(t), F(t))$$

which is over the curve $(x(t), y(t))$.
We find another curve in the space




$$(a + x'(0)t, b + y'(0)t, f(a + x'(0)t, b + y'(0)t))$$

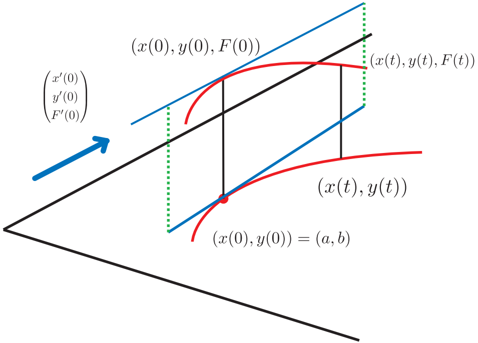
over $(a + x'(0)t, b + y'(0)t)$.
The two curves are tangential at $(a, b, f(a, b))$.

The tangential direction of the curve

$$(x(t), y(t), F(t))$$

at $(x(0), y(0), F(0))$ is

$$\begin{pmatrix} x'(0) \\ y'(0) \\ F'(0) \end{pmatrix}$$



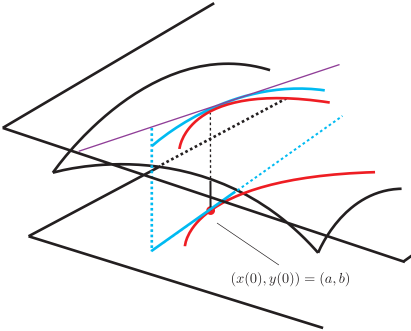
The tangential direction of the curve

$$(a + x'(0)t, b + y'(0)t, G(t))$$

with

$$G(t) = f(a + x'(0)t, b + y'(0)t)$$

is




$$(a + x'(0)t, b + y'(0)t, f(a, b) + f_x(a, b)x'(0)t + f_y(a, b)y'(0)t)$$

# Chain Rule

- The tangential direction of the curve $(x(t), y(t), F(t))$

$$\begin{pmatrix} x'(0) \\ y'(0) \\ F'(0) \end{pmatrix}$$

and that of the other curve $(a + x'(0)t, b + y'(0)t, G(t))$

$$\begin{pmatrix} x'(0) \\ y'(0) \\ f_x(a,b)x'(0) + f_y(a,b)y'(0) \end{pmatrix}$$

are parallel.

- Accordingly we get the identity

$$F'(0) = f_x(a,b)x'(0) + f_y(a,b)y'(0)$$