Binomial Theorem

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Binomial Coefficients

Binomial Coefficients

$$
{}_nC_k = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}
$$

This is the number of the way we choose k balls from n numbered balls:

$$
[1], [2], [3], \dots, [n]
$$

Binomial Theorem

You know well the identities:

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Now how can we develop $(x+y)^4$? We calculate

$$\begin{aligned}(x+y)^4 &= (x+y)(x+y)^3 \\ &= x(x^3 + 3x^2y + 3xy^2 + y^3) + y(x^3 + 3x^2y + 3xy^2 + y^3)\end{aligned}$$

$$\begin{aligned}(x+y)^4 = x^4 &+ 3x^3y + 3x^2y^2 + xy^3 \\ &+ x^3y + 3x^2y^2 + 3xy^3 + y^4 \\ = x^4 &+ 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

Binomial Theorem

General Formula

$$
\begin{aligned}
(x+y)^n &= x^n + {}_nC_1 x^{n-1} y + {}_nC_2 x^{n-2} y^2 + \dots \\
&\quad \dots + {}_nC_{n-k} x^k y^{n-k} + \dots + {}_nC_{n-1} x y^{n-1} + y^n
\end{aligned}
$$

We choose k factors to choose y and the other $(n-k)$ factors are to choose x.

$$
\begin{array}{ccccccc}
 & & 1 & 2 & \dots & n \\
(x+y)^n & = & (x+y) & (x+y) & \dots & (x+y)
\end{array}
$$

Then we get ${}_nC_k x^k y^{n-k}$

A Formula

A Formula

$$ {}_{n+1}C_k = {}_nC_k + {}_nC_{k-1} $$

$$
\begin{array}{rcllllll}
x(x+y)^n & = & x^{n+1} & +{}_nC_1x^ny & +\dots & +{}_nC_kx^{n-k+1}y^k & +\dots \\
y(x+y)^n & = & & x^ny & +\dots & +{}_nC_{k-1}x^{n-k+1}y^k & +\dots \\
\hline
(x+y)^{n+1} & = & x^{n+1} & +{}_{n+1}C_1x^ny & +\dots & +{}_{n+1}C_kx^{n+1-k}y^k & +\dots
\end{array}
$$