Convexity of functions of two varibales (1)

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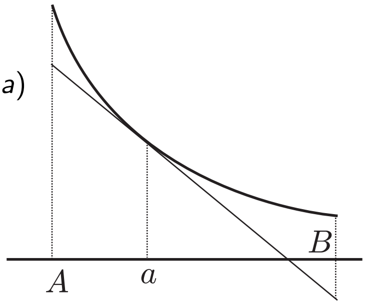
In case of functions of one variable

Given a twice differentaible function $f : (A, B) \longrightarrow \mathbf{R}$ defined on an open subset (A, B). Assume that

$$f''(t) > 0 \quad (t \in (A, B))$$

Then we have an inequality

$$f(t) > f(a) + f'(a)(t - a) \quad (t \neq a)$$



Proof

Define $F(t)$ by $F(t) := f(t) - f(a) - f'(a)(t-a)$. Then it follows that

$$F'(t) = f'(t) - f'(a), \quad F''(t) = f''(t) > 0$$

We recall the fact about increasing functions.

$G'(t) > 0$ $(t \in (A, B))$ とすると

$$A < s < t < B \Rightarrow G(s) < G(t)$$

By the aid of this, we get the inequalities

$$A < s < a < t < B \Rightarrow F'(s) < F'(a) = 0 < F'(t)$$

and the increase abd decrease table

t		a	
F'	$-$	0	$+$
F	$\searrow$	0	$\nearrow$

Then it follows that $F(t) > 0 \quad (t \neq a)$.

Another proof using Taylor's Theorem

Suppose $t \neq a$. Then by Taylor's Theorem, we can find c between t and a satisfying

$$f(t) = f(a) + f'(a)(t-a) + \frac{1}{2}f''(c)(t-a)^2$$

It follows from $f''(c) > 0$ and $(t-a)^2 > 0$ that

$$f(t) > f(a) + f'(a)(t-a)$$

In case of functions of two variables–Directional Derivatives

Given an open subset U in $\mathbf{R}^2$ and a function $f: U \to \mathbf{R}$. Take a point $\mathrm{P}_0(a,b) \in U$ and a non-zero vector $\binom{\xi}{\eta} \neq \vec{0}$ to define a function in t by

$$F(t) = f(a + t\xi, b + t\eta)$$

Then we have with $\mathrm{P}_t(a + t\xi, b + t\eta)$

$$\begin{aligned} F'(t) &= f_x(\mathrm{P}_t)\xi + f_y(\mathrm{P}_t)\eta \\ F''(t) &= \xi\left(f_{xx}(\mathrm{P}_t)\xi + f_{xy}(\mathrm{P}_t)\eta\right) + \eta\left(f_{yx}(\mathrm{P}_t)\xi + f_{yy}(\mathrm{P}_t)\eta\right) \\ &= f_{xx}(\mathrm{P}_t)\xi^2 + 2f_{xy}(\mathrm{P}_t)\xi\eta + f_{yy}(\mathrm{P}_t)\eta^2 \end{aligned}$$

Hesse matrix

We define the Hesse matrix at $P \in U$ by

$$H(f)(P) = \begin{pmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P) \end{pmatrix}$$

- If f is in C^2 class, we have $f_{xy} = f_{yx}$. Thus $H(f)(P)$ is symmetric.
- $F(t) = f(a + t\xi, b + t\eta)$ has the 2nd order derivative in the form

$$F''(t) = \left(H(f)(P_t) \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right)$$

Strictly convex functions of two variables

Given an open convex set U in $\mathbf{R}^2$ and a C^2 function defined on U:

$$f: \; U \longrightarrow \mathbf{R}$$

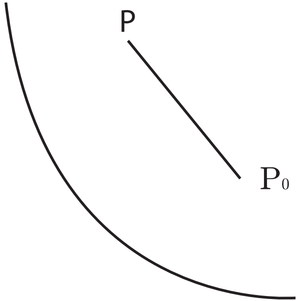
Assume

$$f_{xx}(P) > 0, \; \det\left(H(f)(P)\right) > 0 \; (P \in U)$$

Then for $(x, y) \neq (a, b)$, we have the inequality

$$f(x, y) > f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \quad (x, y) \neq (a, b)$$

Proof (1)



Assume that $\mathrm{P}(x, y)$ and $\mathrm{P}_0(a, b)$ satisfy the condition $\mathrm{P} \neq \mathrm{P}_0$ and define a non-zero vector

$$\vec{v} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} x-a \\ y-b \end{pmatrix} \neq \vec{0}$$

Proof (2)

We apply Taylor's Theorem to $F(t) = f(a + t\xi, b + t\eta)$ to find c with $0 < c < 1$ satisfying

$$F(1) - F(0) = F'(0) \cdot 1 + \frac{1}{2}F''(c)1^2$$

Then it follows that

$$f(x, y) - f(a, b) = f_x(a, b)\xi + f_y(a, b)\eta + \frac{1}{2}(H(f)(P_c)\vec{v}, \vec{v})$$

On the other hand, we have

$$(H(f)(P_c)\vec{v}, \vec{v}) > 0$$

for $\vec{v} \neq \vec{0}$. This implies

$$f(x, y) - f(a, b) > f_x(a, b)\xi + f_y(a, b)\eta$$

Theorem about the positivity of quadratic forms

Theorem

If $a > 0$ and $ab - c^2 > 0$, then

$$ax^2 + 2cxy + by^2 > 0 \quad \left(\begin{pmatrix} x \\ y \end{pmatrix} \neq \vec{0}\right)$$

Proof

$$ax^2 + 2cxy + by^2 = a\left(x + \frac{c}{a}y\right)^2 + \frac{ab - c^2}{a}y^2 \geq 0$$

If $a(x + \frac{c}{a}y)^2 + \frac{ab-c^2}{a}y^2 = 0$, then

$$a(x + \frac{c}{a}y)^2 = \frac{ab - c^2}{a}y^2 = 0 \tag{1}$$

This implies $x = y = 0$. Accordingly we have shown that

$$\begin{pmatrix} x \\ y \end{pmatrix} \neq \vec{0} \quad \Rightarrow ax^2 + 2cxy + by^2 > 0$$