# Lagrange Multiplier (No. 2)

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# Constrained Optimization

Given an open subset $U$ in $\mathbf{R}^2$ and two functions

$$f, g: \ U \to \mathbf{R}$$

**問題**

Maximize or minimize $z = f(x, y)$ subject to $g(x, y) = 0$.

# Review—Theorem

**定理**

Assume that $(a,b) \in U$ satisfies the condition $g(a,b)=0,\ g_y(a,b)\neq 0$. Moreover $f(a,b)$ is a maximal or minimal value of $f(x,y)$ subject to $g(x,y)=0$. Then there exists $\lambda \in \mathbf{R}$ satisfying

$$\left\{\begin{array}{rcrcll} f_x(a,b) & + & \lambda g_x(a,b) & = & 0 & (1) \\ f_y(a,b) & + & \lambda g_y(a,b) & = & 0 & (2) \\ & & g(a,b) & = & 0 & (3) \end{array}\right. \qquad \text{(L)}$$

In the Theorem the identities (1) and (2) are called **Tangency Condition**.

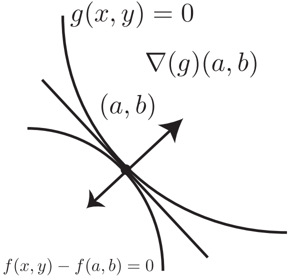
# 接線条件

The Tangency Condition is expressed by

$$\nabla(g)(a,b) = -\lambda \cdot \nabla(f)(a,b)$$

Remark that $\nabla(f)(a,b)$ is the normal vector of the tangent line at $(a,b)$ of the contour of $f(x,y)$:

$$f(x,y) - f(a,b) = 0$$




Sine the two curves $g(x,y)=0$ and $f(x,y)-f(a,b)=0$ share the tangent line at $(a,b)$, they are tangent each othrt.

# Constrained Optimization—Application in Microeconomics

**Example (Utility Maximization)** Let $I, p, q > 0$. We maximaize the utility function

$$u(x, y) = \sqrt{xy}$$

subject to the budget constraint

$$g(x, y) := I - px - qy = 0 \quad (x, y > 0)$$

# Constrained Optimization—Application in Microeconomics(2)

If the utility function is maximal at $(x, y)$ under the budget constraint, there exists $\lambda \in \mathbf{R}$ satisfying

$$\begin{cases} \frac{1}{2} \cdot \frac{\sqrt{y}}{\sqrt{x}} & + & \lambda(-p) & = & 0 & (1) \\ \frac{1}{2} \cdot \frac{\sqrt{x}}{\sqrt{y}} & + & \lambda(-q) & = & 0 & (2) \\ & & I - px - qy & = & 0 & (3) \end{cases}$$

We consider $(1) \times x$ and $(2) \times y$ to get

$$\sqrt{xy} = 2\lambda px = 2\lambda qy$$

Since it follows from (1) that $\lambda \neq 0$, we get

$$px = qy$$

Moreover it follows from (3) that

$$px = qy = \frac{I}{2} \quad \text{thus} \quad x = \frac{I}{2p}, \quad y = \frac{I}{2q}$$

# Constrained Optimization—Application in Microeconomics(3)

$$x(p,q,I) = \frac{I}{2p}, \quad y(p,q,I) = \frac{I}{2q}$$

are called **demand functions**. Moreover Lagrange multiplier obtained from (1)

$$\lambda = \frac{1}{2p} \cdot \frac{\sqrt{\frac{I}{2q}}}{\sqrt{\frac{I}{2p}}} = \frac{1}{2\sqrt{pq}}$$

is called the **Marginal Utulity of Income**.

# Marginal Utility of Income

We define **Indirect Utility** function by

$$v(p,q,I) = u(x(p,q,I), y(p,q,I)) = \sqrt{\frac{I}{2p}} \cdot \sqrt{\frac{I}{2q}} = \frac{I}{2\sqrt{pq}}$$

Then we have

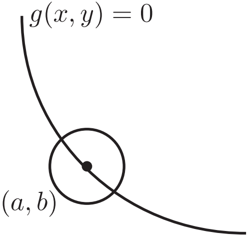
$$\frac{\partial v}{\partial I} = \frac{1}{2\sqrt{pq}} = \lambda(p,q,I)$$

This is the reason why the multiplier $\lambda(p,q,I)$ is called the Marginal Utility of Income. Remark that the identity

$$\frac{\partial v}{\partial I} = \lambda(p,q,I)$$

holds generally.

# Sufficient Condition to be a maximal (minimal) point




Assume

$$g(a,b)=0, \quad g_y(a,b)\neq 0$$

to apply Implicit Function Theorem. We can express the curve $g(x,y)=0$

$$y=\varphi(x)$$

in a neighboehood of $(a,b)$.

If $f$ is minimal or maximal at $(a,b)$ subject to $g(x,y)=0$, the function

$$F(t):=f(t,\varphi(t))$$

satisfies $F'(a)=0$. Moreover if $F'(a)=0$ and

$$F''(a)>0 \quad (\text{resp.} \quad F''(a)<0$$

then $f$ is minimal (resp. maximal) at $(a,b)$ subject to $g(x,y)=0$.

# Solution(1)

We apply Chain Rule to derivate $f(t, \varphi(t))$ and get

$$F'(t) = f_x(t, \varphi(t)) \cdot 1 + f_y(t, \varphi(t)) \cdot \varphi'(t)$$

and

$$\begin{aligned}
F''(t) = f_{xx}(t, \varphi(t)) \cdot 1 + f_{xy}(t, \varphi(t)) \cdot \varphi'(t) \\
+ \varphi'(t) \left( f_{yx}(t, \varphi(t)) \cdot 1 + f_{yy}(t, \varphi(t)) \cdot \varphi'(t) \right) \\
+ f_y(t, \varphi(t)) \cdot \varphi''(t) \\
= f_{xx}(t, \varphi(t)) + 2 f_{xy}(t, \varphi(t)) \cdot \varphi'(t) + f_{yy}(t, \varphi(t)) \cdot \varphi'(t)^2 \\
+ f_y(t, \varphi(t)) \cdot \varphi''(t)
\end{aligned}$$

# Solution(2)

On the other hand we derivate the both sides of $g(t, \varphi(t)) \equiv 0$ by $t$ to get

$$g_x(t, \varphi(t)) \cdot 1 + g_y(t, \varphi(t)) \cdot \varphi'(t) \equiv 0$$

and

$$\begin{aligned} g_{xx}(t, \varphi(t)) + 2g_{xy}(t, \varphi(t)) \cdot \varphi'(t) + g_{yy}(t, \varphi(t)) \cdot \varphi'(t)^2 \\ g_y(t, \varphi(t)) \cdot \varphi''(t) \equiv 0 \end{aligned}$$

# Solution(3)

We let $P_0(a, b)$ and put substitute $t = a$ in the last identities about $g$ to get

$$\varphi'(a) = -\frac{g_x(P_0)}{g_y(P_0)}$$

and

$$\varphi''(a) = -\frac{1}{g_y(a, b)} \left( g_{xx}(P_0) + 2g_{xy}(P_0) \cdot \varphi'(a) + g_{yy}(P_0) \cdot \varphi'(a)^2 \right)$$

# Solution(4)

We put $t = a$ in the identities about $f$ to get

$$\begin{aligned}
F''(a) &= f_{xx}(P_0) + 2f_{xy}(P_0)\cdot\varphi'(a) + f_{yy}(P_0)\cdot\varphi'(a)^2 \\
&\qquad + f_y(P_0)\cdot\varphi''(a) \\
&= f_{xx}(P_0) + 2f_{xy}(P_0)\cdot\varphi'(a) + f_{yy}(P_0)\cdot\varphi'(a)^2 \\
&\qquad - \frac{f_y(P_0)}{g_y(P_0)}\left(g_{xx}(P_0) + 2g_{xy}(P_0)\cdot\varphi'(a) + g_{yy}(P_0)\cdot\varphi'(a)^2\right) \\
&= L_{xx}(P_0,\lambda) + 2L_{xy}(P_0,\lambda)\cdot\varphi'(a) + L_{yy}(P_0,\lambda)\cdot\varphi'(a)^2
\end{aligned}$$

Here we defined $\lambda$ and $L$ by

$$\lambda = -\frac{f_y(P_0)}{g_y(P_0)}, \quad L(x,y,\lambda) = f(x,y) + \lambda\cdot g(x,y)$$

# Solution (5)

Now we eliminate $\varphi'(a)$ in the preceding identities

$$
\begin{aligned}
& F''(a) \\
&= L_{xx}(P_0,\lambda) + 2L_{xy}(P_0,\lambda)\cdot\left(-\frac{g_x(P_0)}{g_y(P_0)}\right) + L_{yy}(P_0,\lambda)\cdot\left(-\frac{g_x(P_0)}{g_y(P_0)}\right)^2 \\
&= \frac{1}{g_y(P_0)^2}\Big(L_{xx}(P_0,\lambda)\cdot g_y(P_0)^2 - 2L_{xy}(P_0,\lambda)\cdot g_x(P_0)g_y(P_0) \\
&\qquad + L_{yy}(P_0,\lambda)\cdot g_x(P_0)^2\Big) \\
&= -\frac{1}{g_y(P_0)^2}\cdot\begin{vmatrix} 0 & g_x(a,b) & g_y(a,b) \\ g_x(a,b) & L_{xx}(a,b,\lambda) & L_{xy}(a,b,\lambda) \\ g_y(a,b) & L_{yx}(a,b,\lambda) & L_{yy}(a,b,\lambda) \end{vmatrix}
\end{aligned}
$$

# Theorem

**Theorem** Assume that there exists $\lambda \in \mathbf{R}$ satisfying

$$\left\{\begin{array}{rcrcll} f_x(a,b) & + & \lambda g_x(a,b) & = & 0 & (1) \\ f_y(a,b) & + & \lambda g_y(a,b) & = & 0 & (2) \\ & & g(a,b) & = & 0 & (3) \end{array}\right. \qquad (\mathrm{L})$$

Moreover if

$$B(a,b,\lambda) := \begin{vmatrix} 0 & g_x(a,b) & g_y(a,b) \\ g_x(a,b) & L_{xx}(a,b,\lambda) & L_{xy}(a,b,\lambda) \\ g_y(a,b) & L_{yx}(a,b,\lambda) & L_{yy}(a,b,\lambda) \end{vmatrix}$$

satisfies $B(a,b,\lambda) < 0$ (resp. $B(a,b,\lambda) > 0$), then $f$ is minimal (resp. maximal) at $(a,b)$ subject to $g(x,y) = 0$. Here we defined the function $L$ by

$$L(x,y,\lambda) := f(x,y) + \lambda g(x,y)$$

# $\varphi''(a)$

$$
\begin{aligned}
\varphi''(a) &= -\frac{1}{g_y(a,b)}\left(g_{xx}(P_0) + 2g_{xy}(P_0)\cdot\varphi'(a) + g_{yy}(P_0)\cdot\varphi'(a)^2\right)\\
&= -\frac{1}{g_y(a,b)^3}\Big(g_{xx}(P_0)\cdot g_y(P_0)^2 - 2g_{xy}(P_0)\cdot g_x(P_0)g_y(P_0)\\
&\qquad + g_{yy}(P_0)\cdot g_x(P_0)^2\Big)\\
&= \frac{1}{g_y(a,b)^3}\begin{vmatrix} 0 & g_x(a,b) & g_y(a,b)\\ g_x(a,b) & g_{xx}(a,b) & g_{xy}(a,b)\\ g_y(a,b) & g_{yx}(a,b) & g_{yy}(a,b)\end{vmatrix}
\end{aligned}
$$