Lagrange Multiplier

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Constrained Optimization

Given an open subset U in $\mathbf{R}^2$ and two functions defined on U

$$f, g: \ U \to \mathbf{R}$$

Problem

Maximize or minimize $z = f(x, y)$ subject to the constraint $g(x, y) = 0$.

Constrained Optimization

Example 1 Maximize or minimize

$z = f(x, y) = 2x + y$ subject to the constraint $g(x, y) = x^2 + y^2 - 1 = 0$

例 **2** Let $I, p, q > 0$. Maximize the utility function

$$u(x, y) = \sqrt{xy}$$

subject to the budget constrint

$$g(x, y) = I - px - qy = 0 \quad (x, y > 0)$$

In this problem, the prices of the 1st good and the second are respectively p and q. It is asked how we maximize the utility by spending the budget I with x units of the 1st good and y units of the second.

Impicit function Theorem

Theorem

Assume that

$$g(a,b)=0, \quad g_y(a,b)\neq 0$$

Then $\{(x,y)\in U;\ g(x,y)=0\}$ is expressed by

$$y=\varphi(x)$$

in a neighborhood of (a,b).

Impicit function Theorem—Example

Take a point (a, b) on the unit circle

$$g(x, y) = x^2 + y^2 - 1 = 0$$

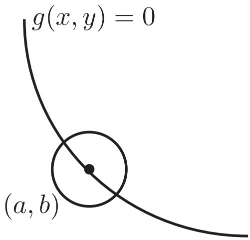
In case $b > 0$

$$y = \sqrt{1 - x^2}$$

In case $b < 0$

$$y = -\sqrt{1 - x^2}$$

Solution (1)



We assume the condition

$$g(a, b) = 0, \quad g_y(a, b) \neq 0$$

to apply Implicity Function Theorem. Then the curve $g(x, y) = 0$ is expressed by

$$y = \varphi(x)$$

in a neighborhood of (a, b).

If f is maximal or minimal at (a, b) subject to $g(x, y) = 0$,

$$F(t) := f(t, \varphi(t))$$

satisfies $F'(a) = 0$.

Solution (2)

We apply Chain Rule to differentiate $F(t)$ by

$$F'(t) = f_x(t, \varphi(t)) \cdot 1 + f_y(t, \varphi(t)) \cdot \varphi'(t)$$

Thus we get

$$0 = F'(a) = f_x(a, b) + f_y(a, b) \cdot \varphi'(a)$$

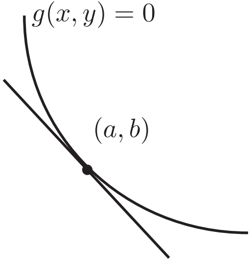
Next we differentiate the both sides of $g(t, \varphi(t)) \equiv 0$ by t to get

$$g_x(t, \varphi(t)) \cdot 1 + g_y(t, \varphi(t)) \cdot \varphi'(t) \equiv 0$$

Then it follows that

$$g_x(a, b) + g_y(a, b) \cdot \varphi'(a) = 0 \quad \textit{i.e.} \quad \varphi'(a) = -\frac{g_x(a, b)}{g_y(a, b)}$$

Another way to find $\varphi'(a)$



The tangent line of the curve $g(x,y)=0$ a (a,b) is

$$g_x(a,b)(x-a)+g_y(a,b)(y-b)=0$$

Then it follows from $g_y(a,b)\neq 0$ that

$$y=-\frac{g_x(a,b)}{g_y(a,b)}(x-a)+b$$

We consider the slope of the tangent line to get

$$\varphi'(a)=-\frac{g_x(a,b)}{g_y(a,b)}$$

Solution (3)

We substitute $\varphi'(a) = -\frac{g_x(a,b)}{g_y(a,b)}$ to $f_x(a,b) + f_y(a,b) \cdot \varphi'(a) = 0$ to get

$$f_x(a,b) - \frac{g_x(a,b)}{g_y(a,b)} \cdot f_y(a,b) = 0$$

Here we define the Lagrange Multiplier by

$$\lambda = -\frac{f_y(a,b)}{g_y(a,b)}$$

Then we find the three identities

$$\left\{ \begin{array}{rcrcl} f_x(a,b) & + & \lambda g_x(a,b) & = & 0 \\ f_y(a,b) & + & \lambda g_y(a,b) & = & 0 \\ & & g(a,b) & = & 0 \end{array} \right. \tag{L}$$

Theorem

Theorem

Assume that $(a, b) \in U$ satisfies the condition $g(a, b) = 0$, $g_y(a, b) \neq 0$. Moreover $f(a, b)$ is a maximum (minimal) value of $f(x, y)$ subject to the constraint $g(x, y) = 0$. Then there exists $\lambda \in \mathbf{R}$ satisfying (L).

Example(1)

Problem Optimize $z = f(x, y) = 2x + y$ subject to the constraint $g(x, y) = x^2 + y^2 - 1 = 0$.

If f is maximal or minimal at (x, y) subject to the constraint $g(x, y) = 0$, there exists $\lambda \in \mathbf{R}$ satisfying

$$\begin{cases} 2 \; + \; \lambda \cdot 2x = 0 & (i) \\ 1 \; + \; \lambda \cdot 2y = 0 & (ii) \\ x^2 + y^2 - 1 = 0 & (iii) \end{cases}$$

If $\lambda = 0$ it follows from (i) that $2 = 0$. Accordingly we find $\lambda \neq 0$. Under this condition, (i) and (ii) imply

$$x = -\frac{1}{\lambda}, \quad y = -\frac{1}{2\lambda} \tag{iv}$$

We substitute these into (iii) to get

Example(2)

$$\frac{1}{\lambda^2} + \frac{1}{4\lambda^2} = 1 \quad \text{thus} \quad \lambda = \pm\frac{\sqrt{5}}{2}$$

Moreover we substitute these into (iv) to find

$$x = \mp\frac{2}{\sqrt{5}}, \quad y = \mp\frac{1}{\sqrt{5}}, \quad \lambda = \pm\frac{\sqrt{5}}{2} \quad \text{(Double Sign Correspond)}$$