Diagonalize $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ by a rotation

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$$\Phi_A(\lambda) = \begin{vmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{vmatrix} = (\lambda-2)^2 - 1$$

$$= (\lambda-3)(\lambda-1).$$

Thus the eigenvalues of A are

$$\lambda = 1, 3.$$

We find eigenvectors.

<u>In case $\lambda = 1$</u>

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Leftrightarrow x + y = 0.$$

The eigenvectors for the eigenval. $\lambda = 1$ are

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

<u>In case $\lambda = 3$</u>

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Leftrightarrow x - y = 0$$

The eigenvectors for $\lambda = 3$ are

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

Choose

$$\vec{r_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{r_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Then

$$A\vec{r_1} = \vec{r_1}, \quad A\vec{r_2} = 3\vec{r_2}$$

and

$$R = (\vec{r_1}\ \vec{r_2}) \text{ is a rotation.}$$

what is more

$$AR = (A\vec{r}_1 \ \ A\vec{r}_3) = (\vec{r}_1 \ \ 3\vec{r}_2)$$

$$= (\vec{r}_1 \ \vec{r}_3)\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = R\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

By multiplying R^{-1} from the left, we get

$$R^{-1}AR = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Supplementary Note: We consider the quadratic form:

$$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = 2x^2 + 2xy + 2y^2.$$

Since R^{-1} is a rotation we get

$$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(R^{-1}A\begin{pmatrix} x \\ y \end{pmatrix}, R^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$$= \left(R^{-1}AR \cdot R^{-1}\begin{pmatrix} x \\ y \end{pmatrix}, R^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

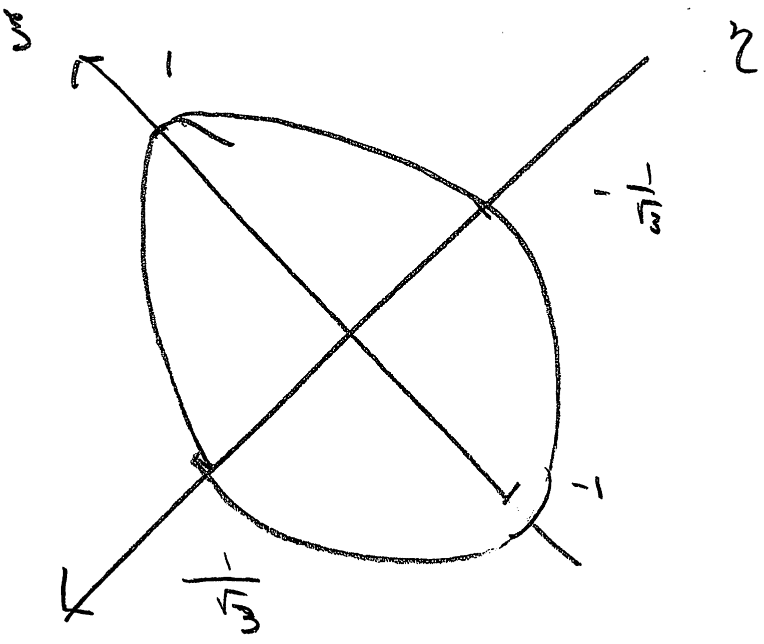
We take the coordinate transformation defined by

$$R^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Then

$$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right)$$

$$= \xi^2 + 3\eta^2$$



$\xi^2 + 3\eta^2 = 1$