Diagonalize $A = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$ by a rotation

We find ~~the~~ eigen polynomial of A by

$$\Phi_A(\lambda) = |\lambda I_2 - A|$$

$$= \begin{vmatrix} \lambda-5 & 2 \\ 2 & \lambda-2 \end{vmatrix} = (\lambda-5)(\lambda-2) - 4$$

$$= \lambda^2 - 7\lambda + 6 = (\lambda-6)(\lambda-1)$$

Thus the eigenvalues of A are $\lambda = 1, 6$. Next we find eigen vectors of A.

<u>In case $\lambda = 1$</u>

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \iff (I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$\iff 2x - y = 0$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (x \neq 0)$$

are the eigenvectors of A for the eigenvalue $\lambda = 1$.

<u>In case $\lambda = 6$.</u>

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 6\begin{pmatrix} x \\ y \end{pmatrix} \iff (6I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$\iff x + 2y = 0$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

are the eigenvectors of A for the eigenvalue $\lambda = 6$

We choose two unit vectors

$$\vec{r_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{r_2} = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Then $R = (\vec{r_1}\ \vec{r_2})$ is a rotation matrix, and

$$AR = (A\vec{r}_1 \ A\vec{r}_2)$$

$$= (\vec{r}_1 \ 6\vec{r}_1)$$

$$= (\vec{r}_1 \ \vec{r}_2)\begin{pmatrix}1 & 0\\ 0 & 6\end{pmatrix} = R\begin{pmatrix}1 & 0\\ 0 & 6\end{pmatrix}$$

We multiply the both hand sides by R^{-1} from the left to get

$$R^{-1}AR = \begin{pmatrix}1 & 0\\ 0 & 6\end{pmatrix}.$$

We consider the quadratic form defined by A:

$$\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = \left(R^{-1}ARR^{-1}\begin{pmatrix}x\\y\end{pmatrix}, R^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$$= \left(\begin{pmatrix}1 & 0\\ 0 & 6\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \begin{pmatrix}\xi\\ \eta\end{pmatrix}\right) = \xi^2 + 6\eta^2$$

Here we used the rotational coordinate transform defined by

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = R^{-1}\begin{pmatrix}x\\y\end{pmatrix}.$$

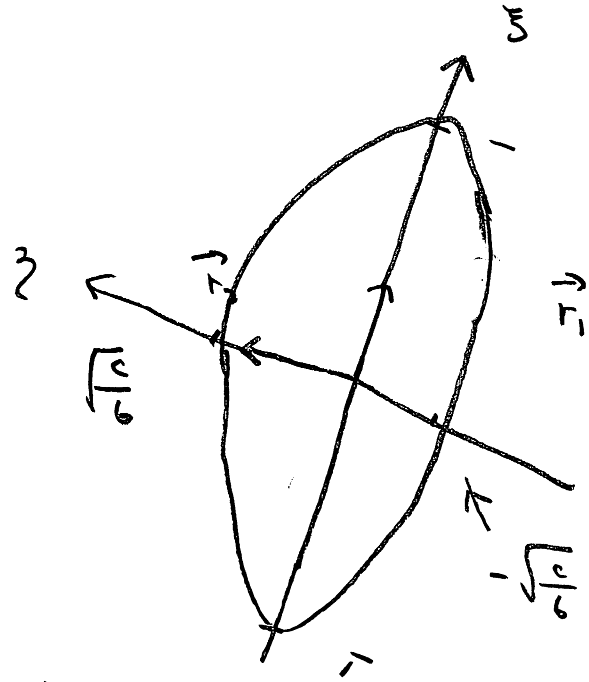
Next we consider Quadratic Curve

$$A\begin{pmatrix}x\\y\end{pmatrix} = c$$

This is expressed in terms of the coordinate system ξ and η by

$$(\#) \qquad \xi^2 + 6\eta^2 = c$$

In case $c > 0$, the quadratic curve (#) is an ellipse:



In case $c = 0$, (#) is the Origine.

In case $c < 0$, (#) is empty.

Diagonalize $A = \begin{pmatrix} -1 & -3 \\ -3 & 7 \end{pmatrix}$ by a rotation.

$$\Phi_A(\lambda) = |\lambda I_2 - A| = \begin{vmatrix} \lambda+1 & 3 \\ 3 & \lambda-7 \end{vmatrix}$$

$$= (\lambda+1)(\lambda-7) - 9 = \lambda^2 - 6\lambda - 16$$

$$= (\lambda-8)(\lambda+2)$$

Thus the eigenvalues of A are $\lambda = -2, 8$. We find eigenvectors of A.

In case $\lambda = -2$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff -x + 3y = 0$$

Accordingly we find the eigenvectors

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3y \\ y \end{pmatrix} = y\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

In case $\lambda = 8$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 8\begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff 3x + y = 0$$

Accordingly we find the eigenvectors

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -3x \end{pmatrix} = x\begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad (x \neq 0)$$

We define two vectors by

$$\vec{p}_1 = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \vec{p}_2 = \frac{1}{\sqrt{10}}\begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Then $P = (\vec{p}_1 \ \vec{p}_2)$ a rotation and we have

$$A\vec{p}_1 = -2\vec{p}_1, \quad A\vec{p}_2 = 8\vec{p}_2.$$

Thus

$$AP = A(\vec{p}_1 \ \vec{p}_2) = (-2\vec{p}_1 \ \ 8\vec{p}_2) = (\vec{p}_1 \ \vec{p}_2)\begin{pmatrix} -2 & 0 \\ 0 & 8 \end{pmatrix}$$

$$= P\begin{pmatrix} -2 & 0 \\ 0 & 8 \end{pmatrix}$$

and A is diagonalized by

$$P^{-1}AP = \begin{pmatrix} -2 & 0 \\ 0 & 8 \end{pmatrix}.$$

We consider the quadratic form defined by A:

$$\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = -x^2-6xy+7y^2$$

Since P^{-1} is a rotation

$$\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = \left(P^{-1}A\begin{pmatrix}x\\y\end{pmatrix}, P^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$$= \left(P^{-1}AP\cdot P^{-1}\begin{pmatrix}x\\y\end{pmatrix}, P^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$$= \left(\begin{pmatrix}-2&0\\0&8\end{pmatrix}\begin{pmatrix}\xi\\\eta\end{pmatrix}, \begin{pmatrix}\xi\\\eta\end{pmatrix}\right) = -2\xi^2+8\eta^2.$$

Here we used the rotational coordinate transformation defined by

$$\begin{pmatrix}\xi\\\eta\end{pmatrix} = P^{-1}\begin{pmatrix}x\\y\end{pmatrix}.$$

Next we consider the quadratic curve

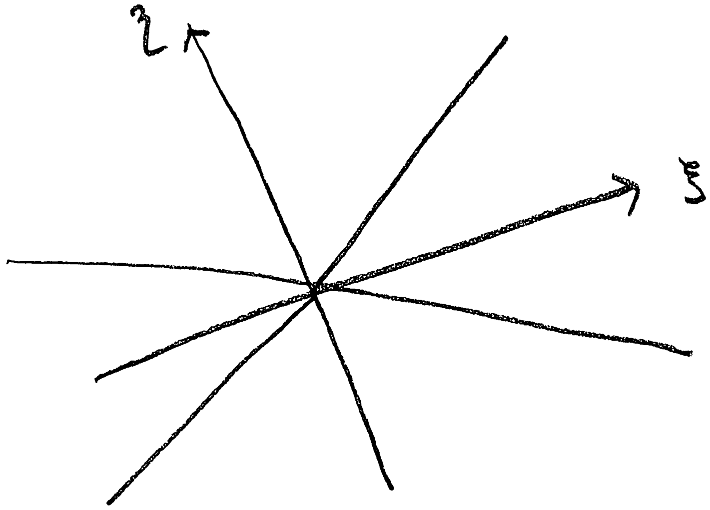
$$(\#) \quad \left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = C$$

This is expressed in terms of the coordinate system (ξ, η) by

$$-2\xi^2+8\eta^2 = C.$$

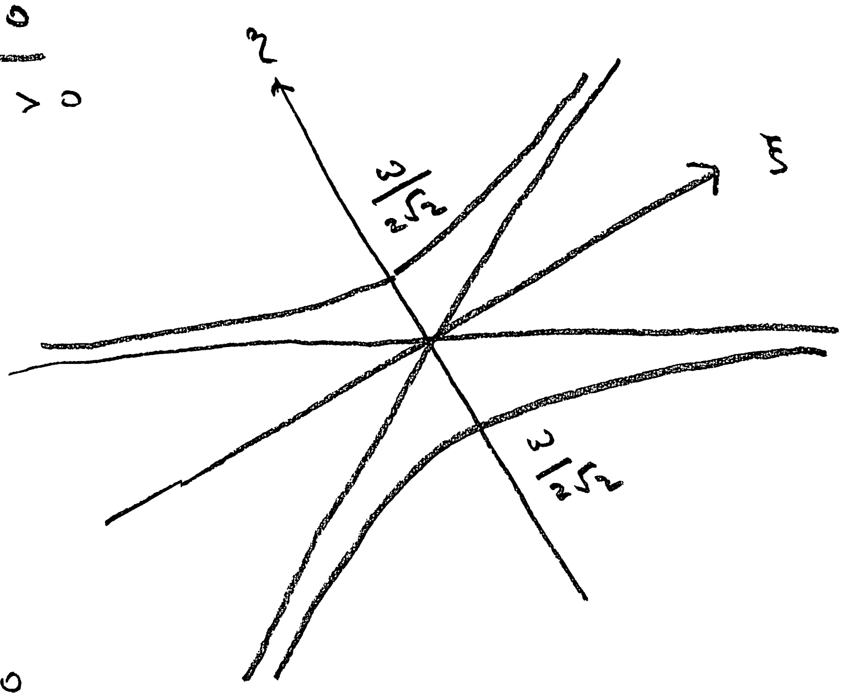
<u>In case $C=0$</u>

$$(\#) \Leftrightarrow \eta = \pm\frac{1}{2}\xi$$



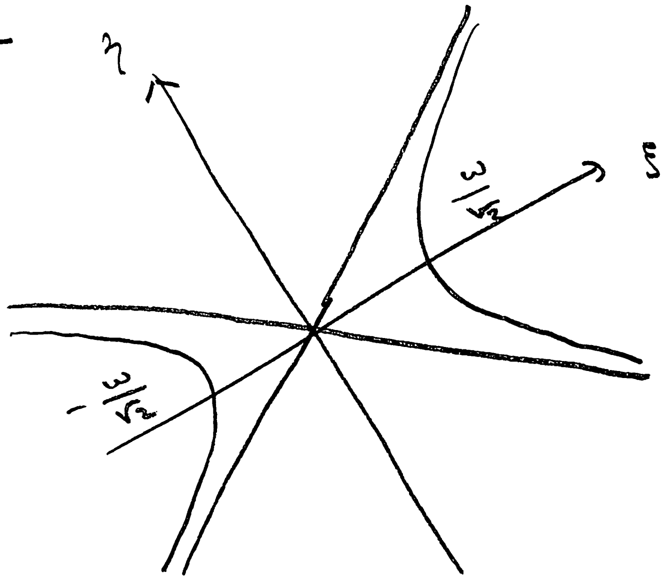
In case $c = \omega^2 > 0$

with $\omega > 0$



In case $c = -\omega^2 < 0$

with $\omega > 0$



Basic Properties of Symmetric Matrices

Given a symmetric matrix

$$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \in M_2(\mathbb{R})$$

Then

$$\Phi_A(\lambda) = \begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix}$$
$$= (\lambda - a)(\lambda - b) - c^2$$
$$= \lambda^2 - (a+b)\lambda + ab - c^2.$$

The discriminant of $\Phi_A(\lambda)$ is

$$D = (a+b)^2 - 4(ab - c^2)$$
$$= (a-b)^2 + 4c^2 \geqslant 0$$

Accordingly $\Phi_A(\lambda)$ has two real roots $\alpha, \beta \in \mathbb{R}$.
What is more, remark that

$$D = 0 \iff a = b \text{ AND } c = 0$$

$$\iff A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a I_2$$

$\boxed{\begin{array}{l}\underline{\text{Theorem 1.}} \text{ Given a symmetric matrix } A \in M_2(\mathbb{R}) \\ \text{Then } \Phi_A(\lambda) \text{ has two real roots. Moreover} \\ \text{if } \Phi_A(\lambda) \text{ has a multiple root, then } A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\end{array}}$

QR分解.

$$\vec{a} = \|\vec{a}\| \vec{p}$$

$$\leftarrow \quad \vec{b} - \vec{w} = \|\vec{b} - \vec{w}\| \vec{q}$$

$$\vec{b} =$$

Theorem 2 Given a symmetric matrix $A \in M_2(\mathbb{R})$. Then A can be diagonalized by a rotation.

Proof. There is nothing to show in case $\Phi_A(x)$ has a multiple root. We may assume that

$$\Phi_A(\lambda) = (\lambda - \alpha)(\lambda - \beta)$$

with $\alpha \neq \beta$. We take eigenvectors $\vec{q}_1, \vec{q}_2 \in \mathbb{R}^2$ satisfying

$$A\vec{q}_1 = \alpha \vec{q}_1, \quad A\vec{q}_2 = \beta \vec{q}_2, \quad \vec{q}_1 \neq \vec{0}, \quad \vec{q}_2 \neq \vec{0}.$$

Moreover define

$$\vec{p}_1 = \frac{1}{\|\vec{q}_1\|}\vec{q}_1, \quad \vec{p}_2 = \frac{1}{\|\vec{q}_2\|}\vec{q}_2.$$

Then

$$\|\vec{p}_1\| = \|\vec{p}_2\| = 1, \quad (\vec{p}_1, \vec{p}_2) = 0$$

$$A\vec{p}_1 = \alpha \vec{p}_1, \quad A\vec{p}_2 = \beta \vec{p}_2.$$

Then $(\vec{p}_1 \ \vec{p}_2)$ or $(\vec{p}_1 \ -\vec{p}_2)$ is a rotation $R = (\vec{r}_1 \ \vec{r}_2)$, which satisfies

$$AR = (A\vec{r}_1 \ A\vec{r}_2) = (\alpha\vec{r}_1 \ \beta\vec{r}_2)$$
$$= (\vec{r}_1 \ \vec{r}_2)\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = R\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

Accordingly

$$R^{-1}AR = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

<u>Remark</u> Take the determinant of

$$R^{-1}AR = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Then

$$|A| = |R^{-1}AR| = \alpha\beta.$$

Thus we have equivalence

$$|A| \begin{matrix} > \\ = \\ < \end{matrix} 0 \iff \alpha\beta \begin{matrix} > \\ = \\ < \end{matrix} 0.$$

We consider the quadratic curve

$$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = C,$$

which is expressed in terms of the coordinate system

$$(\#_1) \qquad \begin{pmatrix} \xi \\ \eta \end{pmatrix} = R^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$$

by

$$(\#_2) \qquad \alpha\xi^2 + \beta\eta^2 = C$$

(i) In case $\alpha, \beta > 0$

$$(\#_2) \text{ is } \begin{cases} \text{an ellipse} & (C > 0) \\ \text{the origin} & (C = 0) \\ \text{empty} & (C < 0) \end{cases}$$

(ii) In case $\alpha, \beta < 0$

$$(\#_2) \text{ is } \begin{cases} \text{empty} & (c > 0) \\ \text{the origin} & (c = 0) \\ \text{an ellipse} & (c < 0) \end{cases}$$

(iii) In case $\alpha\beta < 0$. ($\Leftrightarrow D < 0$)

$$(\#_2) \text{ is } \begin{cases} \text{two lines intersecting at the origin} & c = 0 \\ \text{hyperbola.} & c \neq 0 \end{cases}$$

The following theorem 3 is important.

<u>Theorem 3.</u> Given a symmetric $A \in M_2(\mathbb{R})$ $\left(A = \begin{pmatrix} a & c \\ c & e \end{pmatrix}\right)$
Then the following conditions
(i), (ii), (iii) are equivalent.

(i) $\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) > 0 \quad \left(\begin{pmatrix} x \\ y \end{pmatrix} \neq \vec{0}\right)$

(ii) The eigenvalues of A α, β are positive:

$$\alpha, \beta > 0$$

(iii) $|A| > 0, \quad a > 0$