$$A = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$$

Find the eigen polynomial of A

$$\Phi_A(\lambda) = |\lambda I_2 - A| = \overbrace{\begin{vmatrix} \lambda-5 & 2 \\ 2 & \lambda-2 \end{vmatrix}}^{\lambda I_2 - A} = \cdots = (\lambda-6)(\lambda-1)$$

The eigen values of A are $\lambda = 1, 6$.

$\underline{\lambda = 1}$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 1\cdot\begin{pmatrix} x \\ y \end{pmatrix} \iff (I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$-4x + 2y = 0$

$\updownarrow$

$\downarrow$

$2x - y = 0$

$$\iff 2x - y = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (x \neq 0)$$

$\underline{\lambda = 6}$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 6\begin{pmatrix} x \\ y \end{pmatrix} \iff (6I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff \begin{pmatrix} \boxed{1 \quad 2} \\ 2 \quad 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$\rightarrow x + 2y = 0$

$\rightarrow 2x + 4y = 0$

$$\iff x + 2y = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

$$\vec{r}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix} \qquad x=5 \longrightarrow \qquad A\vec{r}_1 = 1\cdot\vec{r}_1$$

$$\vec{r}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix} \qquad \longrightarrow \qquad A\vec{r}_2 = 6\,\vec{r}_2$$

$$R = (\vec{r}_1\ \vec{r}_2) = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}$$ A rotation

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$$AR = (A\vec{r}_1\ \ A\vec{r}_2) = (\vec{r}_1\ \ 6\vec{r}_2) \underset{\uparrow}{=} (\vec{r}_1\ \vec{r}_2)\begin{pmatrix}1 & 0\\0 & 6\end{pmatrix}$$

$$= R\begin{pmatrix}1 & 0\\0 & 6\end{pmatrix}$$

$$\xrightarrow[R^{-1}\cdot]{} \qquad R^{-1}AR = \begin{pmatrix}1 & 0\\0 & 6\end{pmatrix}$$

Diagonalized by a rotation.

Quadratic form defined by A $\qquad A = \begin{pmatrix}5 & -2\\-2 & 2\end{pmatrix}$

$$\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = 5x^2 \underset{\substack{\| \\ 2\times(-2)}}{-4}\,xy + 2y^2$$

R^{-1} is also a rotation!!.

$5x^2 - 4xy + 2y^2$

$$\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = \left(R^{-1}A\begin{pmatrix}x\\y\end{pmatrix}, R^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$$= \left(R^{-1}AR\cdot R^{-1}\begin{pmatrix}x\\y\end{pmatrix}, R^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$$= \left(\begin{pmatrix}1&0\\0&6\end{pmatrix}\begin{pmatrix}\xi\\\eta\end{pmatrix}, \begin{pmatrix}\xi\\\eta\end{pmatrix}\right)$$

$$= \xi^2 + 6\eta^2$$

$$\begin{pmatrix}\xi\\\eta\end{pmatrix} = R^{-1}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix}x\\y\end{pmatrix} = R\begin{pmatrix}\xi\\\eta\end{pmatrix}$$

$$= \xi\vec{r_1} + \eta\vec{r_2}$$

diagonal form of

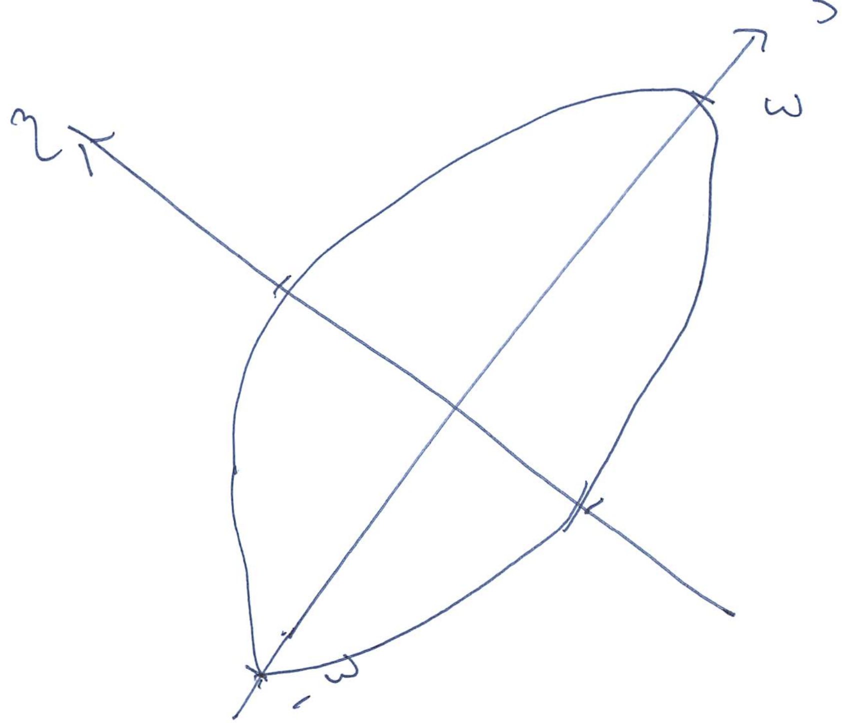
$$\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right)$$

Quadratic Curve

$$\left(A \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) = c$$

$$\Longleftrightarrow \quad \xi^2 + 6\eta^2 = c,$$

In case $c = \omega^2 > 0$ with $\omega > 0$.



$\xi^2 + 6\eta^2 = \omega^2$

$\eta = 0$

$\to \xi^2 = \omega^2$

$\to \xi = \pm\omega$

$\xi = 0 \to$

$\eta^2 = \dfrac{\omega^2}{6}.$

$\to \eta = \pm\dfrac{\omega}{\sqrt{6}}$

In case $C = 0$. $\xi^2 + 6\eta^2 = 0 \Leftrightarrow \xi = \eta = 0$

$$\boxed{\begin{array}{l} \beta_1, \beta_2 \geq 0 \\ \beta_1 + \beta_2 = 0 \end{array} \Longrightarrow \beta_1 = \beta_2 = 0}$$

$\xi^2 = 6\eta^2 = 0 \rightarrow \xi = \eta = 0$.

In case $C < 0$ $\quad \underset{\displaystyle 0}{\underset{\vee\!\!\mid}{\xi^2 + 6\eta^2}} = \underset{\displaystyle 0}{\underset{\wedge}{C}}$ is empty.

$$A = \begin{pmatrix} -1 & -3 \\ -3 & 7 \end{pmatrix}$$

$$\lambda I_2 - A$$

$$\Phi_A(\lambda) = |\lambda I_2 - A| = \left| \boxed{\begin{matrix} \lambda+1 & 3 \\ 3 & \lambda-7 \end{matrix}} \right| = \cdots = (\lambda-8)(\lambda+2)$$

<u>$\lambda = -2$</u>

$$A\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix} \iff (2I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$-x + 3y = 0$$

$$\iff \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$3x - 9y = 0$$

$$\iff -x + 3y = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3y \\ y \end{pmatrix} = y\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

<u>$\lambda = 8$</u>

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 8\begin{pmatrix} x \\ y \end{pmatrix} \iff (8I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$\iff \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff 3x + y = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -3x \end{pmatrix} = x\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$\vec{r}_1 = \frac{1}{\sqrt{10}}\begin{pmatrix}3\\1\end{pmatrix}$ $\qquad A\vec{r}_1 = -2\vec{r}_1$

$\vec{r}_2 = \frac{1}{\sqrt{10}}\begin{pmatrix}-1\\3\end{pmatrix}$ $\qquad R = (\vec{r}_1\ \ \vec{r}_2)$

$\searrow \quad x = -\frac{1}{\sqrt{10}}. \qquad A\vec{r}_2 = 8\vec{r}_2$

$R = \frac{1}{\sqrt{10}}\begin{pmatrix}3 & -1\\1 & 3\end{pmatrix}$ is a rotation

$$AR = (A\vec{r}_1\ \ A\vec{r}_2) = (-2\vec{r}_1\ \ 8\vec{r}_2)$$

$$= (\vec{r}_1\ \ \vec{r}_2)\begin{pmatrix}-2 & 0\\0 & 8\end{pmatrix}$$

$$= R\begin{pmatrix}-2 & 0\\0 & 8\end{pmatrix}$$

$R^{-1}\cdot$

$$R^{-1}AR = \begin{pmatrix}-2 & 0\\0 & 8\end{pmatrix}$$

$(\ A(\vec{x}\ \ \vec{y}) = (A\vec{x}\ \ A\vec{y})$

$(\vec{a}_1\ \ \vec{a}_2)\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha\vec{a}_1 + \beta\vec{a}_2$

Diagonalization by a rotation

Quadratic form

$$A = \begin{pmatrix} -1 & -3 \\ -3 & 7 \end{pmatrix}$$

$$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = -x^2 \underbrace{-6}_{=\,2\times(-3)} xy + 7y^2$$

$$= \left(R^{-1}A\begin{pmatrix} x \\ y \end{pmatrix}, R^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

R^{-1} is a rotation

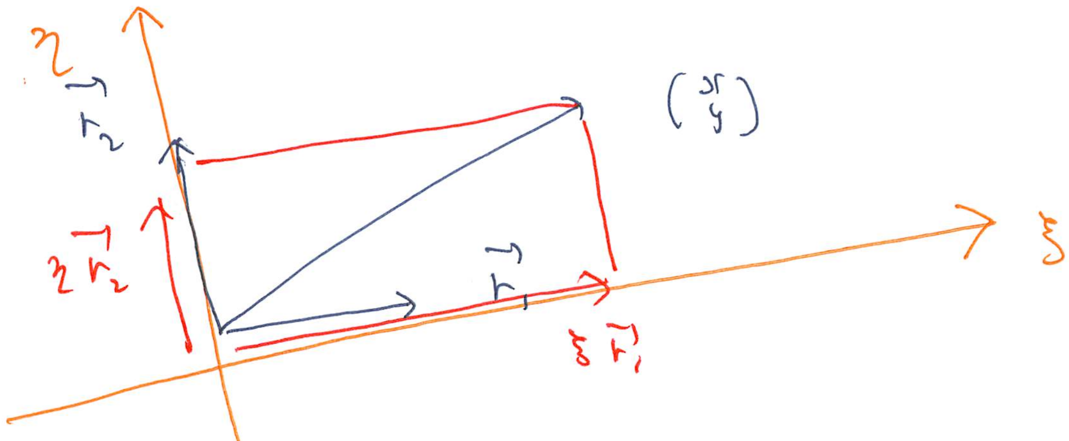
$$= \left(R^{-1}A \underbrace{R\cdot R^{-1}}_{=\,I_2}\begin{pmatrix} x \\ y \end{pmatrix}, R^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\begin{pmatrix} -2 & 0 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right)$$

$$= -2\xi^2 + 8\eta^2$$

Diagonal form of $\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right)$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = R^{-1}\begin{pmatrix} x \\ y \end{pmatrix} \longleftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = R\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \xi\vec{r}_1 + \eta\vec{r}_2$$

$$\vec{r}_1 = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \vec{r}_2 = \frac{1}{\sqrt{10}}\begin{pmatrix} -1 \\ 3 \end{pmatrix}$$



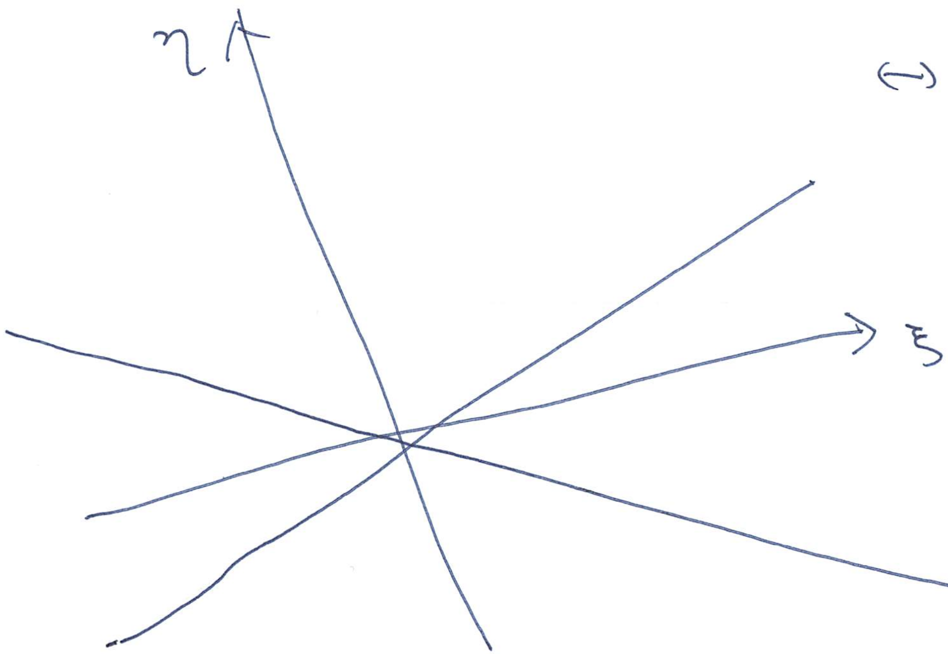
The quadratic curve $\left(A\binom{x}{y}, \binom{x}{y}\right) = c$

$$\Leftrightarrow \quad -2\xi^2 + 8\eta^2 = c$$

In case $c = 0$

$$-2\xi^2 + 8\eta^2 = 0 \Leftrightarrow \eta^2 = \frac{1}{4}\xi^2$$

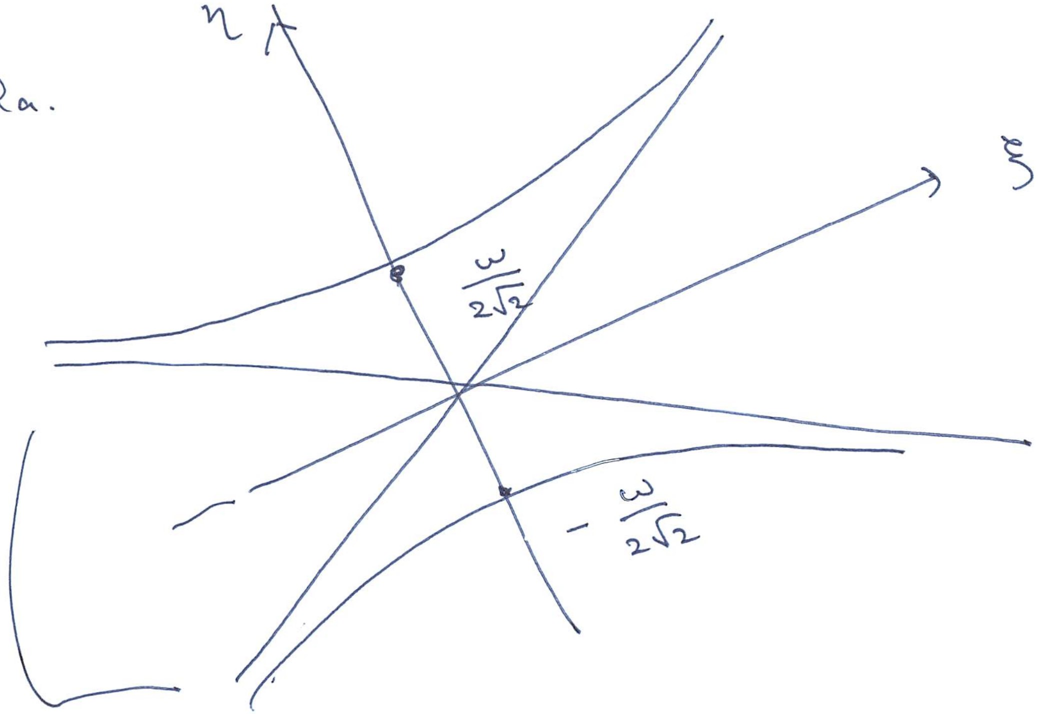
$$\Leftrightarrow \quad \eta = \pm\frac{1}{2}\xi$$



In case $c = \omega^2 > 0$ with $\omega > 0$

$$-2\xi^2 + 8\eta^2 = \omega^2$$

hyperbola.



Asymptotic Lines

$\eta = 0$

$\rightarrow \; -2\xi^2 = \omega^2$

$\quad\;\; \wedge \qquad \vee$

$\quad\;\; 0 \qquad 0$

$\xi = 0 \quad 8\eta^2 = \omega^2$

$$\eta = \frac{\pm 1}{2\sqrt{2}}\,\omega$$

In case $c = -\omega^2 < 0$ with $\omega > 0$

$$-2\xi^2 + \delta\eta^2 = -\omega^2$$