Diagonalize A by a rotation matrix.

We find ~~the eigen~~ polynomial of A by

$$\Phi_A(\lambda) = |\lambda I_2 - A|$$

$$= \begin{vmatrix} \lambda-5 & 2 \\ 2 & \lambda-2 \end{vmatrix} = (\lambda-5)(\lambda-2)-4$$

$$= \lambda^2 - 7\lambda + 6 = (\lambda-6)(\lambda-1)$$

Thus the eigenvalues of A are $\lambda = 1, 6$. Next we find eigenvectors of A.

<u>In case $\lambda = 1$</u>

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \iff (I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$\iff 2x - y = 0$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (x \neq 0)$$

are the eigenvectors of A for the eigenvalue $\lambda = 1$.

<u>In case $\lambda = 6$.</u>

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 6\begin{pmatrix} x \\ y \end{pmatrix} \iff (6I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$\iff x + 2y = 0$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

are the eigenvectors of A for the eigenvalue $\lambda = 6$

We choose two unit vectors

$$\vec{r_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{r_2} = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Then $R = (\vec{r_1}\ \vec{r_2})$ is a rotation matrix, and

$$AR = (A\vec{r_1} \;\; A\vec{r_2})$$

$$= (\vec{r_1} \;\; 6\vec{r_1})$$

$$= (\vec{r_1} \;\; \vec{r_2}) \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} = R \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

We multiply the both hand sides by R^{-1} from the left to get

$$R^{-1}AR = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}.$$

Supplementary Note : We consider the quadratic form defined by A :

$$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(R^{-1}ARR^{-1}\begin{pmatrix} x \\ y \end{pmatrix}, R^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$$= \left(\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) = \xi^2 + 6\eta^2$$

Here we used the rotational coordinate transform defined by

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = R^{-1}\begin{pmatrix} x \\ y \end{pmatrix}.$$