I (1) $\Phi_A(\lambda) = |\lambda I_2 - A| = \begin{vmatrix} \lambda+1 & 3 \\ 3 & \lambda-7 \end{vmatrix}$

$$= (\lambda+1)(\lambda-7) - 9 = \lambda^2 - 6\lambda - 16$$
$$= (\lambda-8)(\lambda+2)$$

Thus the eigenvalues of A are $\lambda = -2, 8$. We find eigenvectors of A.

<u>In case $\lambda = -2$</u>

$$A\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff -x+3y=0$$

Accordingly we find the eigenvectors

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3y \\ y \end{pmatrix} = y\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

<u>In case $\lambda = 8$</u>

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 8\begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff 3x+y=0$$

Accordingly we find the eigenvectors

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -3x \end{pmatrix} = x\begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad (x \neq 0).$$

We define two vectors by

$$\vec{P_1} = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \vec{P_2} = \frac{1}{\sqrt{10}}\begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Then $P = (\vec{P_1}\ \vec{P_2})$ a rotation and we have

$$A\vec{P_1} = -2\vec{P_1}, \quad A\vec{P_2} = 8\vec{P_2}.$$

Thus

$$AP = A(\vec{P_1}\ \vec{P_2}) = (-2\vec{P_1}\ \ 8\vec{P_2}) = (\vec{P_1}\ \vec{P_2})\begin{pmatrix} -2 & 0 \\ 0 & 8 \end{pmatrix}$$
$$= P\begin{pmatrix} -2 & 0 \\ 0 & 8 \end{pmatrix}$$

and A is diagonalized by

$$P^{-1}AP = \begin{pmatrix} -2 & 0 \\ 0 & 8 \end{pmatrix}.$$

We consider the quadratic form defined by A:

$$\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = -x^2 + 2xy + 7y^2$$

Since P^{-1} is a rotation

$$\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = \left(P^{-1}A\begin{pmatrix}x\\y\end{pmatrix}, P^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$$= \left(P^{-1}AP \cdot P^{-1}\begin{pmatrix}x\\y\end{pmatrix}, P^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$$= \left(\begin{pmatrix}-2 & 0\\0 & 8\end{pmatrix}\begin{pmatrix}\xi\\\eta\end{pmatrix}, \begin{pmatrix}\xi\\\eta\end{pmatrix}\right) = -2\xi^2 + 8\eta^2$$

Here we used the rotational coordinate transformation defined by

$$\begin{pmatrix}\xi\\\eta\end{pmatrix} = P^{-1}\begin{pmatrix}x\\y\end{pmatrix}.$$

(2) $\Phi_A(\lambda) = |\lambda I_2 - A| = \begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{vmatrix}$

$= (\lambda-1)^2 - 2^2 = (\lambda+1)(\lambda-3).$

Thus the eigenvectors of A are $\lambda = -1, 3$. We find the eigenvectors as follows.

In case $\lambda = -1$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff x + y = 0.$$

Thus the eigenvectors are

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix} = x\begin{pmatrix} 1 \\ -1 \end{pmatrix} \ (x \neq 0).$$

In case $\lambda = 3$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \iff x - y = 0$$

Thus the eigenvectors are

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad x \neq 0.$$

We define two vectors

$$\vec{p_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } \vec{p_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which satisfy

$$A\vec{p_1} = -\vec{p_1} \text{ and } A\vec{p_2} = 3\vec{p_2}.$$

Moreover $P = (\vec{p_1}\ \vec{p_2})$ is a rotation and

$$AP = (A\vec{p_1}\ A\vec{p_2}) = (-\vec{p_1}\ \ 3\vec{p_2}) = (\vec{p_1}\ \vec{p_2})\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= P\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Accordingly we find that A is diagonalized by

$$P^{-1}AP = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}.$$

We consider the quadratic form defined by A:

$$\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = x^2+4xy+y^2.$$

Since P^{-1} is a rotation, we get

$$\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = \left(P^{-1}AP\cdot P^{-1}\begin{pmatrix}x\\y\end{pmatrix}, P^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$$= \left(\begin{pmatrix}-1 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}\xi\\\eta\end{pmatrix}, \begin{pmatrix}\xi\\\eta\end{pmatrix}\right) = -\xi^2+3\eta^2$$

Here we used the rotational coordinate transformation defined by

$$\begin{pmatrix}\xi\\\eta\end{pmatrix} = P^{-1}\begin{pmatrix}x\\y\end{pmatrix}.$$

(3)

$$\Phi_A(\lambda) = |\lambda I_2 - A| = \begin{vmatrix} \lambda-1 & -4 \\ -4 & \lambda-7 \end{vmatrix}$$

$$= (\lambda-1)(\lambda-7)-16 = \lambda^2 - 8\lambda - 9$$

$$= (\lambda+1)(\lambda-9).$$

Thus the eigenvectors of A are $\lambda = -1, 9$. We find the eigenvectors as follows.

<u>In case $\lambda = -1$</u>

$$A\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} -2 & -4 \\ -4 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Leftrightarrow x + 2y = 0$$

Thus the eigenvectors are

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

<u>In case $\lambda = 9$</u>

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 9\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Leftrightarrow 2x - y = 0$$

Thus the eigenvectors are

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (x \neq 0).$$

We two eigenvectors by

$$\vec{P_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \vec{P_2} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

so that

$$P = (\vec{P_1}\ \vec{P_2}) = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

is a rotation. Moreover

$$A\vec{P_1} = -\vec{P_1} \quad \text{and} \quad A\vec{P_2} = 9\vec{P_2}$$

which leads us to the identity

$$AP = (A\vec{P_1}\ A\vec{P_2}) = (-\vec{P_1}\ 9\vec{P_2})$$

$$= (\vec{P_1}\ \vec{P_2})\begin{pmatrix} -1 & 0 \\ 0 & 9 \end{pmatrix} = P\begin{pmatrix} -1 & 0 \\ 0 & 9 \end{pmatrix}.$$

Accordingly we have

$$P^{-1} A P = \begin{pmatrix} -1 & 0 \\ 0 & 9 \end{pmatrix}$$

We consider the quadratic form defined by A:

$$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = x^2 + 8xy + 7y^2.$$

Since P^{-1} is a rotation we get

$$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(P^{-1} A P \cdot P^{-1}\begin{pmatrix} x \\ y \end{pmatrix}, P^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$$= \left(\begin{pmatrix} -1 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) = -\xi^2 + 9\eta^2$$

Here we used the rotational coordinate transformation defined by

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = P^{-1}\begin{pmatrix} x \\ y \end{pmatrix}.$$

II.

$$\Phi_{P^{-1}AP}(\lambda) = |\lambda I_2 - P^{-1}AP|$$

$$= |P^{-1}\lambda I_2 P - P^{-1}AP|$$

$$= |P^{-1}(\lambda I_2 - A)P|$$

$$= |P^{-1}| \cdot |\lambda I_2 - A| \cdot |P|.$$

We remark that

$$|P^{-1}| \cdot |P| = |P^{-1}P| = |I_2| = 1.$$

Thus

$$\Phi_{P^{-1}AP} = |P^{-1}| \cdot |P| \; \Phi_A(\lambda)$$

$$= 1 \cdot \Phi_A(\lambda) = \Phi_A(\lambda)$$

III Since $\vec{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \neq \vec{0}$, $p_j \neq 0$ for some j.

Then $c\vec{p} = \begin{pmatrix} \vdots \\ cp_j \\ \vdots \end{pmatrix} = \vec{0}$ implies that $cp_j = 0$.

It follows that $c = 0$.

IV (1) $A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$ (2) $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$

(3) $\begin{pmatrix} 1 & -3 \\ -3 & 4 \end{pmatrix}$

V.

$$R_\theta R_{\theta'} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\theta' - \sin\theta\sin\theta' & -\cos\theta\sin\theta' - \sin\theta\cos\theta' \\ \sin\theta\cos\theta' + \cos\theta\sin\theta' & -\sin\theta\sin\theta' + \cos\theta\cos\theta' \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta+\theta') & -\sin(\theta+\theta') \\ \sin(\theta+\theta') & \cos(\theta+\theta') \end{pmatrix}$$

$$= R_{\theta+\theta'}.$$

We use the above identity to show that

$$R_\theta R_{-\theta} = R_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$$R_{-\theta} R_\theta = R_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

Thus by definition of the inverse we get

$$R_\theta^{-1} = R_{-\theta}$$

VI First remark that

$$ {}^tQ = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = Q. $$

Thus

$$ {}^tQ\,Q = Q\,{}^tQ = Q^2 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}^2 $$

$$ = \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} $$

$$ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2. $$

By a basic property of the transposition we get

$$ (Q\vec{v}, Q\vec{w}) = (\vec{v}, {}^tQ\,Q\,\vec{w}) $$

$$ = (\vec{v}, I_2\vec{w}) = (\vec{v}, \vec{w}) $$