$$A = \begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix}.$$

$$|\lambda I_2 - A| = \left| \boxed{\begin{matrix} \lambda - 3 & 2 \\ 4 & \lambda - 1 \end{matrix}} \right| = (\lambda - 3)(\lambda - 1) - 8 = \lambda^2 - 4\lambda - 5$$
$$= (\lambda - 5)(\lambda + 1)$$

$$(\lambda I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff A\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigenvalues of A are $\lambda = -1, 5$.

<u>In case $\lambda = -1$</u>

$$A\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix} \iff (-I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$\iff 2x - y = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \end{pmatrix}. \quad (x \neq 0)$$

are the eigenvectors of A for the eigenvalue $\lambda = -1$.

In case $\lambda = 5$.

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix} \iff (5I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$\iff x + y = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

are the eigenvectors of A for the eigenvalue $\lambda = 5$.

$$\vec{p_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{p_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$A\vec{p_1} = -\vec{p_1}, \quad A\vec{p_2} = 5\vec{p_2}$$

We define

$$P = (\vec{p_1} \ \vec{p_2}) = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$|P| = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 1 + 2 = 3 \neq 0. \quad P \text{ is regular.}$$

$$AP = \begin{pmatrix} A\vec{p}_1 & A\vec{p}_2 \end{pmatrix} = \begin{pmatrix} -\vec{p}_1 & 5\vec{p}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \vec{p}_1 & \vec{p}_2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} = P \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}.$$

$$\longrightarrow \quad P^{-1} A P = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}.$$

Definitions

Let A be a 2×2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then $\alpha \in \mathbf{R}$ is called an eigenvalue of A if there exists $\begin{pmatrix} x \\ y \end{pmatrix} \neq \vec{0}$ satisfying

$$(\#) \quad A \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} x \\ y \end{pmatrix}$$

In this situation $\begin{pmatrix} x \\ y \end{pmatrix} \neq \vec{0}$ satisfying $(\#)$ is called an eigen vector Of A for the eigenvalue α.

Remarks

Remark that

$$(\#) \quad A \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} x \\ y \end{pmatrix} \quad \Leftrightarrow \quad (\#)' \quad (\alpha I_2 - A) \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

and that

$$B\vec{v} = \vec{0} \quad \text{for some } \vec{v} \neq \vec{0} \quad \Leftrightarrow \quad |B| = 0.$$

Accordingly it follows that

$$\alpha \text{ is an eigen value of } A \quad \Leftrightarrow \quad |\alpha I_2 - A| = 0$$

Eigenpolynomial

$a+d = tr(A)$ trace of A

We define the eigenpolynomial of A by

$$\Phi_A(\lambda) := |\lambda I_2 - A| = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix}$$
$$= \lambda^2 - (a+d)\lambda + ad - bc \; = |A|$$

Remark that

$$\alpha \text{ is an eigenvalue of } A \quad \Leftrightarrow \quad \Phi_A(\alpha) = 0$$

Theorem

Theorem

We assume that $\Phi_A(\lambda)$ is factorized by

$$\Phi_A(\lambda) = (\lambda - \alpha)(\lambda - \beta)$$

with $\alpha, \beta \in \mathbf{R}$ satisfying $\underline{\alpha \neq \beta}$. Moreover we assume that the two vectors $\vec{p}_1, \vec{p}_2$ satisfy

$$A\vec{p}_1 = \alpha\vec{p}_1, \quad A\vec{p}_2 = \beta\vec{p}_2$$

with $\vec{p}_1 \neq \vec{0}$, $\vec{p}_2 \neq \vec{0}$. Then

$$P = (\vec{p}_1 \; \vec{p}_2)$$

is regular.

c_2

Proof

Remark that

$|P| \neq 0 \Leftrightarrow$ P is regular $\Leftrightarrow$ $\left(c_1\vec{p}_1 + p_2\vec{p}_2 = \vec{0} \Rightarrow c_1 = c_2 = 0\right)$.

We assume that

c_2

$$c_1\vec{p}_1 + p_2\vec{p}_2 = \vec{0}$$

and multiply the both hand sides by $(\beta I_2 - A)$ to get

$$c_1(\beta - \alpha)\vec{p}_1 = \vec{0} \longrightarrow c_1 \vec{p}_1 = \vec{0}.$$

It follwos from $\beta - \alpha \neq 0$ and $\vec{p}_1 \neq \vec{0}$ that

$$c_1 = 0$$

In this situation we get

$$c_2\vec{p}_2 = \vec{0}$$

It follows from $\vec{p}_2 \neq \vec{0}$ that

$$c_2 = 0. \leftarrow 0$$

$${}^tA = {}^t\left(\begin{pmatrix}2\\2\end{pmatrix} \begin{pmatrix}2\\5\end{pmatrix}\right) = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} = A$$

Example (1)

Let $A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$. Then $= |\lambda I_2 - A| = \left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} \right|$

$$\Phi_A(\lambda) = \begin{vmatrix} \lambda - 2 & -2 \\ -2 & \lambda - 5 \end{vmatrix} = (\lambda - 1)(\lambda - 6)$$

Thus the eigenvalues of A are $\lambda = 1, 6$. We find the eigenvectors of A as follows.

<u>In case $\lambda = 1$</u> $\Leftarrow (I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Leftrightarrow x + 2y = 0$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

are the eigen$\boxed{\text{values}}$ of A for the eigenvalue $\lambda = 1$.

vectors.

Example (2)

<u>In case $\lambda = 6$</u>

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 6\begin{pmatrix} x \\ y \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \quad \Leftrightarrow \quad 2x - y = 0.$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad (x \neq 0)$$

are the eigen~~values~~ of A for the eigenvalue $\lambda = 6$.
vectors

Example (3)

We choose tow vectors $\vec{q}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\vec{q}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and also a matrix

$$Q = (\vec{q}_1\ \vec{q}_2) = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

The Q is regular by the Theorem proven above and

$$\begin{aligned} AQ &= (A\vec{q}_1\ A\vec{q}_2) = (\vec{q}_1\ 6\vec{q}_2) \\ &= (\vec{q}_1\ \vec{q}_2)\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} = Q\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \end{aligned}$$

Thus we can diagonalize A by

$$Q^{-1}AQ = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

Symmetric matrices

Remark that

$$\vec{q}_1 \perp \vec{q}_2$$

This is not a coincidence. Actually we have the following theorem.

Theorem

Let A be a 2×2 matirx and assume that A is symmetric *i.e.* ${}^tA = A$. Moreover we assume that

$$A\vec{p} = \alpha\vec{p}, \quad A\vec{q} = \alpha\vec{q}$$

with $\alpha, \beta \in \mathbf{R}$ satisfying $\alpha \neq \beta$. Then

β

$$\vec{p} \perp \vec{q}$$

proof

$${}^tA = A.$$

$$\downarrow$$

$$(A\vec{p}, \vec{q}) = (\vec{p}, {}^tA\vec{q}) = (\vec{p}, A\vec{q})$$

Moreover

$$(A\vec{p}, \vec{q}) = (\alpha\vec{p}, \vec{q}) = \alpha(\vec{p}, \vec{q})$$

$$(\vec{p}, A\vec{q}) = (\vec{p}, \beta\vec{q}) = \beta(\vec{p}, \vec{q})$$

These two identities lead us to

$$(\alpha - \beta)(\vec{p}, \vec{q}) = 0$$

$$\alpha \neq \beta \quad \longrightarrow \quad (\vec{p}, \vec{q}) = 0.$$

Why do we need symmetric matrices?

We are given a symmetric matrix $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$. We introduce the quadratice form for A by

$$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = ax^2 + 2cxy + by^2$$

Quadratic form defined by A.

In case $A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$, we have

$$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = 2x^2 + 4xy + 5y^2$$

We need a more detailed analysis about the diagonalization of symmetric matrices for its application.

Rotation Matrices(1)

We define the rotation matrix of the angle θ by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The rotation can be figured out by the identity

$$R_\theta \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} \cos(\alpha+\theta) \\ \sin(\alpha+\theta) \end{pmatrix}$$

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Rotation matrices

$|R_\theta| = 1$

Moreover we find that

$$R_\theta^{-1} = \frac{1}{|R_\theta|}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = {}^tR_\theta$$

Accordingly

$${}^tR_\theta \cdot R_\theta = R_\theta \cdot {}^tR_\theta = I_2$$

This identity leads us to

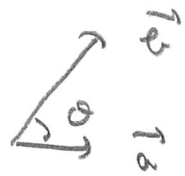
$$(R_\theta\vec{v}, R_\theta\vec{w}) = (\vec{v}, \vec{w})$$

for any $\vec{v}, \vec{w} \in \mathbf{R}^2$. In fact

$$LHS = (\vec{v}, {}^tR_\theta R_\theta\vec{w}) = (\vec{v}, I_2\vec{w}) = (\vec{v}, \vec{w})$$

$\vec{a}, \vec{e} \in \mathbb{R}^2$

$(\vec{a}, \vec{e}) = \|\vec{a}\| \cdot \|\vec{e}\| \cos\theta.$



Diagonalization of symmetric matrices

Let us go back to the example $A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$. We can diagonalize A by a rotation matrix chosen as folloes. We define two unit vectors

$$\vec{r}_1 = \frac{1}{\sqrt{5}}\boxed{(2 \quad -1)}, \quad \vec{r}_2 = \frac{1}{\sqrt{5}}(1 \quad 2)$$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Then $R = (\vec{r}_1 \; \vec{r}_2)$ is a rotation matrix and we have

$\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$A\vec{r}_1 = \vec{r}_1, \quad A\vec{r}_2 = 6\vec{r}_2$$

Accordingly

$$AR = (A\vec{r}_1 \; A\vec{r}_2) = (\vec{r}_1 \; 6\vec{r}_2) = (\vec{r}_1 \; \vec{r}_2)\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} = R\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

Thus we have deduced $R^{-1}AR = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$

Diagonalization of symmetric matrices

We now consider the quadratic form of A. Remark that R^{-1} is a rotation. It follwos from this fact that

$$\left(A\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\right)=\left(R^{-1}A\begin{pmatrix}x\\y\end{pmatrix},R^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$$=\left(R^{-1}A\boxed{R\cdot R^{-1}}\begin{pmatrix}x\\y\end{pmatrix},R^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$$=\left(\begin{pmatrix}1&0\\0&6\end{pmatrix}\begin{pmatrix}\xi\\\eta\end{pmatrix},\begin{pmatrix}\xi\\\eta\end{pmatrix}\right)=\xi^2+6\eta^2$$

$= 2x^2+4xy + 5y^2$

R^{-1} is a rotation

Here we used a rotaional coordinate transformation defined by

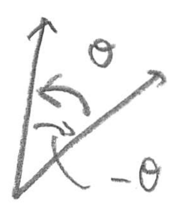
$$\begin{pmatrix}\xi\\\eta\end{pmatrix}=R^{-1}\begin{pmatrix}x\\y\end{pmatrix}$$

$R^{-1}AR = \begin{pmatrix}1&0\\0&6\end{pmatrix}$

$$R = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

$$R^{-1} = {}^tR = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

$$= \begin{pmatrix}\cos(-\theta) & -(\sin(-\theta))\\ \sin(-\theta) & \cos(-\theta)\end{pmatrix}$$



Diagonalize $A = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$

if possible by a rotation matrix.