p 2016/12/20

I

$$|\lambda I_3 - A| = \begin{vmatrix} \lambda-3 & -1 & -1 \\ -1 & \lambda-3 & -1 \\ -1 & -1 & \lambda-3 \end{vmatrix} = \begin{vmatrix} \lambda-2 & 2-\lambda & 0 \\ -1 & \lambda-3 & -1 \\ -1 & -1 & \lambda-3 \end{vmatrix}$$

$R_1 \to R_1 + (-1) R_2$

$$= (\lambda-2) \begin{vmatrix} 1 & -1 & 0 \\ -1 & \lambda-3 & -1 \\ -1 & -1 & \lambda-3 \end{vmatrix}$$

$$= (\lambda-2) \begin{vmatrix} 1 & -1 & 0 \\ 0 & \lambda-4 & -1 \\ 0 & -2 & \lambda-3 \end{vmatrix}$$

$$= (\lambda-2) \begin{vmatrix} \lambda-4 & -1 \\ -2 & \lambda-3 \end{vmatrix}$$

$$= (\lambda-2)(\lambda^2-7\lambda+10) = (\lambda-2)^2(\lambda-5)$$

$\lambda I_3 - A$ is not regular iff $\lambda = 2$ or $\lambda = 5$

II The orthogonal projection of $\vec{q}$ along $\vec{p}$ is

$$\vec{w} = \frac{(\vec{p}, \vec{q})}{\|\vec{p}\|^2} \vec{p} = \frac{2}{2} \vec{p} = \vec{p}$$

Then $\vec{p}$ and $\vec{q} - \vec{w}$ is orthogonal:

$$(\vec{p}, \vec{q} - \vec{w}) = 0$$

$$\vec{q} - \vec{w} = \vec{q} - \vec{p} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}.$$

We normalize the two vectors $\vec{p}$ and $\vec{q} - \vec{w}$ by

$$\vec{r}_1 = \frac{1}{\|\vec{p}_1\|} \vec{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{r}_2 = \frac{1}{\|\vec{q} - \vec{w}\|} (\vec{q} - \vec{w}) = \frac{1}{3} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}.$$

Then $\vec{r}_1, \vec{r}_2$ are an orthonormal basis of V.

III

$$|\lambda I_2 - A| = \begin{vmatrix} \lambda-3 & 2 \\ 4 & \lambda-1 \end{vmatrix} = (\lambda-3)(\lambda-1) - 8$$
$$= \lambda^2 - 4\lambda - 5 = (\lambda-5)(\lambda+1).$$

Thus the eigenvalue of A are $\lambda = -1, 5$.

<u>In case $\lambda = -1$</u>

$$(\#) \quad A\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix} \iff (-I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$\iff \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff 2x - y = 0$$

we put $y = t$ to get the solution to $(\#)$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2t \\ t \end{pmatrix} = t\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

<u>In case $\lambda = 5$.</u>

$$(\#) \quad A\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix} \iff (5I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$\iff \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff x + y = 0$$

We put $x = t$ to get the solution to $(\#)$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -t \end{pmatrix} = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We define two vectors by

$$\vec{p_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \vec{p_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Then we have the identities

$$A\vec{p_1} = -\vec{p_1}, \quad A\vec{p_2} = 5\vec{p_2}.$$

We define a matrix P by $P = (\vec{p_1} \ \vec{p_2})$

Then it follows that

$$
\begin{aligned}
AP &= (A\vec{p_1} \;\; A\vec{p_2}) \\
&= (-\vec{p_1} \;\; 5\vec{p_2}) = (\vec{p_1} \;\; \vec{p_2}) \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \\
&= P \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}
\end{aligned}
$$

What is more, P is regular since

$$
|P| = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3 \neq 0.
$$

Accordingly the matrix A is diagonalized by

$$
P^{-1}AP = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}
$$