2016/12/21

I

$$\vec{w} = \frac{(\vec{p}, \vec{q})}{\|\vec{p}\|^2} \vec{p} = \frac{-8}{10} \vec{p} = -\frac{4}{5} \vec{p}$$

II (1) The orthogonal projection of $\vec{q}$ along $\vec{p}$ is given by

$$\vec{w} := \frac{(\vec{p}, \vec{q})}{\|\vec{p}\|^2} \vec{p} = \frac{2}{4} \vec{p} = \frac{1}{2} \vec{p}.$$

We remark that

$$\vec{q} - \vec{w} = \vec{q} - \frac{1}{2}\vec{p} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

is orthogonal to $\vec{p}$. Then we find that

$$\vec{r}_1 = \frac{1}{\|\vec{p}\|} \vec{p} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{r}_2 = \frac{1}{\|\vec{q} - \vec{w}\|} (\vec{q} - \vec{w}) = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

is an orthonormal basis of V. Moreover the orthogonal projection of $\vec{c}$ along V is given by

$$\vec{w}_2 := (\vec{r}_1, \vec{c})\, \vec{r}_1 + (\vec{r}_2, \vec{c})\, \vec{r}_2$$

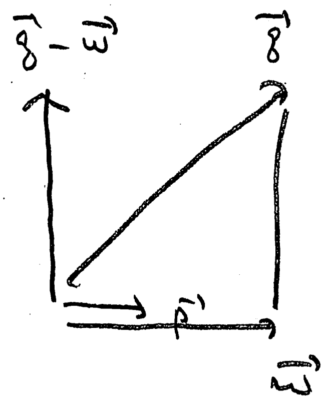
$$= \left(\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) \cdot \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$+ \left(\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) \cdot \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

(2) The orthogonal projection of $\vec{q}$ along $\vec{p}$ is given by

$$\vec{w} := \frac{(\vec{p}, \vec{q})}{\|\vec{p}\|^2}\, \vec{p} = \frac{8}{4}\, \vec{p} = 2\vec{p}$$



Remark that

$$\vec{q} - \vec{w} = \vec{q} - 2\vec{p} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

is orthogonal to $\vec{p}$. Then we find that

$$\vec{r}_1 = \frac{1}{\|\vec{p}\|}\, \vec{p} = \frac{1}{2}\vec{p} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{r}_2 = \frac{1}{\|\vec{q} - \vec{w}\|}(\vec{q} - \vec{w}) = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

is an orthonormal basis of V. Moreover the orthogonal projection of $\vec{c}$ along V is given by

$$\vec{w}_2 = (\vec{r}_1, \vec{c})\, \vec{r}_1 + (\vec{r}_2, \vec{c})\, \vec{r}_2$$

$$= \left(\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) \cdot \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$+ \left(\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

II

(1) We evaluate $|\lambda I_2 - A|$ by

$$\Phi_A(\lambda) := |\lambda I_2 - A|$$

$$= \begin{vmatrix} \lambda-2 & -2 \\ -2 & \lambda-5 \end{vmatrix} = (\lambda-2)(\lambda-5) - 4$$

$$= \lambda^2 - 7\lambda + 6 = (\lambda-1)(\lambda-6)$$

to find that the eigenvalues of A are

$\lambda = 1, 6$. We find the eigenvectors of A.

$\underline{\lambda = 1}$ $\quad A\begin{pmatrix} x \\ y \end{pmatrix} = 1\cdot\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow (I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{o}$

$$\Leftrightarrow \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{o} \Leftrightarrow x + 2y = 0$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

are the eigen vectors of A for the e.v. $\lambda = 1$.

$\underline{\lambda = 6}$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 6\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow (6I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{o}$$

$$\Leftrightarrow \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{o} \Leftrightarrow 2x - y = 0$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (x \neq 0)$$

are the eigenvectors of A for the e.v. $\lambda = 6$

We define two vectors by

$$\vec{p_1} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \vec{p_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and a matrix

$$P = (\vec{p_1}\ \vec{p_2}) = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then P is regular since

$$|P| = \begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} = -5 \neq 0.$$

Moreover we have

$$AP = (A\vec{p_1}\ A\vec{p_2}) = (\vec{p_1}\ 6\vec{p_1})$$

$$= (\vec{p_1}\ \vec{p_2}) \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} = P \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

and accordingly

$$P^{-1} A P = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}.$$

(2) We evaluate $|\lambda I_2 - A|$ by

$$\Phi_A(\lambda) := \begin{vmatrix} \lambda+1 & -3 \\ -3 & \lambda-7 \end{vmatrix} = (\lambda+1)(\lambda-7) - 9$$

$$= \lambda^2 - 6\lambda - 16$$

$$= (\lambda - 8)(\lambda + 2)$$

to find that the eigenvalues of A are

$$\lambda = -2,\ 8.$$

We find the eigenvectors of A as follows.

$\underline{\lambda = -2}$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix} \iff (-2I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$\iff \begin{pmatrix} -1 & -3 \\ -3 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff x + 3y = 0$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3y \\ y \end{pmatrix} = y\begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

are the eigenvectors of A for the e.val. $\lambda = -2$.

$\underline{\lambda = 8}$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 8\begin{pmatrix} x \\ y \end{pmatrix} \iff (8I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$\iff \begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff 3x - y = 0.$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 3x \end{pmatrix} = x\begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (x \neq 0)$$

are the eigenvectors of A for the e.val. $\lambda = 8$.

We choose two vectors by

$$\vec{p_1} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \quad \vec{p_2} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

and define a matrix

$$P = (\vec{p_1}\ \vec{p_2}) = \begin{pmatrix} -3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Then P is regular since

$$|P| = \begin{vmatrix} -3 & 1 \\ 1 & 3 \end{vmatrix} = -10 \neq 0.$$

Moreover we have

$$AP = (A\vec{p_1}\ A\vec{p_2}) = (-2\vec{p_1}\ \ 8\vec{p_2})$$

$$= (\vec{p_1}\ \vec{p_2}) \begin{pmatrix} -2 & 0 \\ 0 & 8 \end{pmatrix} = P \begin{pmatrix} -2 & 0 \\ 0 & 8 \end{pmatrix}$$

and accordingly

$$P^{-1}AP = \begin{pmatrix} -2 & 0 \\ 0 & 8 \end{pmatrix}.$$

(3) We evaluate $|\lambda I_2 - A|$ by

$$\Phi_A(\lambda) := \begin{vmatrix} \lambda-1 & -4 \\ -4 & \lambda-7 \end{vmatrix} = (\lambda-1)(\lambda-7) - 16$$

$$= \lambda^2 - 8\lambda - 9 = (\lambda-9)(\lambda+1)$$

to find that the eigenvalues of A are

$$\lambda = -1,\ 9.$$

We find the eigenvectors of A as follows.

$\underline{\lambda = -1}$ $A\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow (-I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$

$$\Leftrightarrow \begin{pmatrix} -2 & -4 \\ -4 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Leftrightarrow x + 2y = 0$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

are the eigenvectors of A for the e. val. $\lambda = -1$.

$\underline{\lambda = 9}$ $A\begin{pmatrix} x \\ y \end{pmatrix} = 9\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow (9I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$

$$\Leftrightarrow \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Leftrightarrow 2x - y = 0.$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (x \neq 0)$$

are the eigenvectors of A for the e. val.

$\lambda = 9$.

We choose two vectors

$$\vec{p_1} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \vec{p_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and define a matrix

$$P = (\vec{p_1} \ \vec{p_2}) = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

Then P is regular since

$$|P| = \begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} = -5 \neq 0$$

Moreover we have

$$\begin{aligned} AP &= (A\vec{p_1} \ A\vec{p_2}) \\ &= (-\vec{p_1} \ 9\vec{p_2}) = (\vec{p_1} \ \vec{p_2}) \begin{pmatrix} -1 & 0 \\ 0 & 9 \end{pmatrix} \\ &= P \begin{pmatrix} -1 & 0 \\ 0 & 9 \end{pmatrix} \end{aligned}$$

and accordingly

$$P^{-1}AP = \begin{pmatrix} -1 & 0 \\ 0 & 9 \end{pmatrix}$$

(4) We evaluate $|\lambda I_2 - A|$ by

$$\Phi_A(\lambda) = \begin{vmatrix} \lambda-1 & -2 \\ -3 & \lambda+4 \end{vmatrix} = (\lambda-1)(\lambda+4) - 6$$

$$= \lambda^2 + 3\lambda - 10 = (\lambda+5)(\lambda-2)$$

to find that the eigenvalues of A are

$$\lambda = -5, 2.$$

We find eigenvectors of A as follows.

$\underline{\lambda = -5}$ $A\begin{pmatrix} x \\ y \end{pmatrix} = -5\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow (5I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$

$$\Leftrightarrow \begin{pmatrix} -6 & -2 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Leftrightarrow 3x + y = 0$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -3x \end{pmatrix} = x\begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad (x \neq 0)$$

are the eigenvectors of A for the e. val. $\lambda = -5$.

$\underline{\lambda = 2}$ $A\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow (2I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$

$$\Leftrightarrow \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Leftrightarrow x - 2y = 0$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ y \end{pmatrix} = y\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

are the eigenvectors of A for the e. val. $\lambda = 2$.

We choose two vectors

$$\vec{p}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad \vec{p}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and define a matrix

$$P = (\vec{p}_1 \ \vec{p}_2) = \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}.$$

Then P is regular since

$$|P| = \begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} = 7 \neq 0.$$

Moreover we have

$$\begin{aligned} AP &= (A\vec{p}_1 \ A\vec{p}_2) \\ &= (-5\vec{p}_1 \ 2\vec{p}_2) = (\vec{p}_1 \ \vec{p}_2)\begin{pmatrix} -5 & 0 \\ 0 & 2 \end{pmatrix} \\ &= P\begin{pmatrix} -5 & 0 \\ 0 & 2 \end{pmatrix} \end{aligned}$$

and accordingly

$$P^{-1}AP = \begin{pmatrix} -5 & 0 \\ 0 & 2 \end{pmatrix}.$$

(5) We evaluate $|\lambda I_2 - A|$ by

$$\Phi_A(\lambda) := \begin{vmatrix} \lambda+4 & 2 \\ -3 & \lambda-1 \end{vmatrix} = (\lambda+4)(\lambda-1)+6$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1)$$

Thus the eigenvalues of A are $\lambda = -2, -1$.

We find next the eigenvectors of A.

$\underline{\lambda = -2}$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow (-2I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$\Leftrightarrow \begin{pmatrix} 2 & 2 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Leftrightarrow x + y = 0$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

are the eigenvectors for e. val. $\lambda = -2$.

$\underline{\lambda = -1}$ $\quad A\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow (-I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$

$$\Leftrightarrow \begin{pmatrix} 3 & 2 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Leftrightarrow 3x + 2y = 0.$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}y \\ y \end{pmatrix} = \frac{1}{3}y\begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (y \neq 0)$$

are the eigenvalues for e. val. $\lambda = -1$.

We choose two vectors by

$$\vec{p}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \vec{p}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

and define a matrix

$$P = (\vec{p}_1 \ \vec{p}_2) = \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix}.$$

Then P is regular since

$$|P| = \begin{vmatrix} -1 & -2 \\ 1 & 3 \end{vmatrix} = -1 \neq 0.$$

Moreover we have

$$AP = (A\vec{p}_1 \ A\vec{p}_2)$$

$$= (-2\vec{p}_1 \ -\vec{p}_2) = (\vec{p}_1 \ \vec{p}_2) \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= P \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

and accordingly

$$P^{-1}AP = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

IV

$$R_1 R_2 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 & -\cos\theta_1 \sin\theta_2 - \sin\theta_1 \cos\theta_2 \\ \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2 & -\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \end{pmatrix}$$

Since $|R_1| = \cos^2\theta_1 + \sin^2\theta_1 = 1$

$$R_1^{-1} = \frac{1}{1} \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $|A| = ad - bc \neq 0$,

~~then~~

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$