Gram-Schmidt Orthogonalization

Nobuyuki TOSE

December 20, 2016

Problem

We are given two vectors

$$\vec{p}, \vec{q} \in \mathbf{R}^n \quad \text{satisfying} \quad \vec{p} \nparallel \vec{q}$$

We define a subset in $\mathbf{R}^n$ by

$$V := \{x\vec{p} + y\vec{q} \in \mathbf{R}^n;\ x, y \in \mathbf{R}\}$$

called the linear subspace spaned by $\vec{p}$ and $\vec{q}$. We are given another vector $\vec{c} \in \mathbf{R}^n$. The problem is to find $\vec{c}_0 \in V$ satisfying

$$\vec{c} - \vec{c}_0 \perp V \quad \text{i.e.} \quad (\vec{c} - \vec{c}_0, \vec{v}) = 0 \quad (\vec{v} \in V)$$

Remark that this condition is equivalent to

$$(\vec{c} - \vec{c}_0, \vec{p}) = (\vec{c} - \vec{c}_0, \vec{q}) = 0$$

Orthnormal Basis (1)

We take the orthogonal projection of $\vec{q}$ to the direction of $\vec{q}$:

$$\vec{w} = \frac{(\vec{p}, \vec{q})}{||\vec{p}||^2} \cdot \vec{p}$$

In this situation we have

$$\vec{q} - \vec{w} \perp \vec{p}$$

Moreover

$$\vec{q} - \vec{w} \neq \vec{0}$$

In fact if $\vec{q} = \vec{w} = *\vec{p}$, then $\vec{p} \parallel \vec{q}$. This contradicts the hypothesis. We define two vectors

$$\vec{r}_1 = \frac{1}{||\vec{p}||}\vec{p}, \quad \vec{r}_2 = \frac{1}{||\vec{q} - \vec{w}||}(\vec{q} - \vec{w})$$

called an orthonormal basis of V.

Orthnormal Basis (2)

The orthonormal basis $\vec{r}_1$ and $\vec{r}_2$ enjoys the basic property

$$||\vec{r}_1|| = ||\vec{r}_2|| = 1, \quad (\vec{r}_1, \vec{r}_2) = 0$$

We can express the vector $\vec{c}_0 \in V$ by

$$\vec{c}_0 = \xi\vec{r}_1 + \eta\vec{r}_2$$

Moreover $(\vec{c} - \vec{c}_0, \vec{r}_1) = (\vec{c} - \vec{c}_0, \vec{r}_2) = 0$ implies

$$0 = (\vec{c} - \vec{c}_0, \vec{r}_1) = (\vec{c} - \xi\vec{r}_1 - \eta\vec{r}_2, \vec{r}_1) = (\vec{c}, \vec{r}_1) - \xi$$

$$0 = (\vec{c} - \vec{c}_0, \vec{r}_2) = (\vec{c} - \xi\vec{r}_1 - \eta\vec{r}_2, \vec{r}_2) = (\vec{c}, \vec{r}_2) - \eta$$

Accordingly we find that

$$\vec{c}_0 = (\vec{c}, \vec{r}_1)\vec{r}_1 + (\vec{c}, \vec{r}_2)\vec{r}_2$$