Eigenvalue Problems

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Review

Theorem

For $B \in M_2(\mathbf{R})$, the following conditions are equivalent:

- **(i)** B is regular.
- **(ii)** $B\vec{v} = \vec{0} \Rightarrow \vec{v} = \vec{0}$
- **(iii)** $|B| \neq 0$

Theorem

For $B \in M_2(\mathbf{R})$, the following conditions are equivalent:

- **NOT(i)** B is not regular (singular).
- **NOT(ii)** There exists $\vec{v} \neq \vec{0}$ satisfying $B\vec{v} = \vec{0}$.
- **NOT(iii)** $|B| = 0$

Example (1)

Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. We consider the system of equations:

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

First remark that the system is equivalent to

$$(\#) \quad (\lambda I_2 - A) \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}, \quad \text{where} \quad \lambda I_2 - A = \begin{pmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{pmatrix}$$

We also remark that

$$|\lambda I_2 - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{vmatrix} = (\lambda + 1)(\lambda - 5)$$

Example (2)

In case $\lambda \neq -1, 5$ $(\#) \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$

In case $\lambda = -1$

$$(\#) \Leftrightarrow \begin{pmatrix} -2 & -2 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Leftrightarrow x + y = 0$$

We put $y = t$ to get the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Example (3)

In case $\lambda = 5$

$$(\#) \Leftrightarrow \begin{pmatrix} 4 & -2 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Leftrightarrow 2x - y = 0$$

We put $y = t$ to get the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ 2t \end{pmatrix} = t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Diagonalization (1)

We define two vectors

$$\vec{p}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \vec{p}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and a 2×2 matrix

$$P = (\vec{p}_1 \ \vec{p}_2) = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$$

The matrix P is regular since

$$|P| = \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = -3 \neq 0$$

Diagonalization (2)

$$\begin{aligned} AP &= A(\vec{p}_1\ \vec{p}_2) = (A\vec{p}_1\ A\vec{p}_2) \\ &= (-\vec{p}_1\ 5\vec{p}_2) \\ &= (\vec{p}_1\ \vec{p}_2)\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} = P\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \end{aligned}$$

We multiply P^{-1} from the left to get

$$P^{-1}AP = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

This process is called a diagonalization of A. To see how it dows mean, we introduce the coordinate transform given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \xi\vec{p}_1 + \eta\vec{p}_2 = (\vec{p}_1\ \vec{p}_2)\begin{pmatrix} \xi \\ \eta \end{pmatrix} = P\begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

Diagonalization (3)

In this situation the map

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

can be clarified by using the coordinate transform as follows.

$$\begin{aligned} \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} &= P^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} \\ &= P^{-1} A \begin{pmatrix} x \\ y \end{pmatrix} \\ &= P^{-1} A P \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \end{aligned}$$

Definitions

Let A be a 2×2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then $\alpha \in \mathbf{R}$ is called an eigenvalue of A if there exists $\begin{pmatrix} x \\ y \end{pmatrix} \neq \vec{0}$ satisfying

$$(\#) \quad A \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} x \\ y \end{pmatrix}$$

In this situation $\begin{pmatrix} x \\ y \end{pmatrix} \neq \vec{0}$ satisfying $(\#)$ is called an eigen vector 0f A for the eigenvalue α.

Remarks

Remark that

$$(\#) \quad A\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} x \\ y \end{pmatrix} \;\Leftrightarrow\; (\#)' \quad (\alpha I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

and that

$$B\vec{v} = \vec{0} \quad \text{for some } \vec{v} \neq \vec{0} \quad \Leftrightarrow \quad |B| = 0.$$

Accordingly it follows that

$$\alpha \text{ is an eigen value of } A \quad \Leftrightarrow \quad |\alpha I_2 - A| = 0$$

Eigenpolynomial

We define the eigenpolynomial of A by

$$\Phi_A(\lambda) := |\lambda I_2 - A| = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix}$$
$$= \lambda^2 - (a+d)\lambda + ad - bc$$

Remark that

$$\alpha \text{ is an eigenvalue of } A \quad \Leftrightarrow \quad \Phi_A(\alpha) = 0$$

Theorem

Theorem

We assume that $\Phi_A(\lambda)$ is factorized by

$$\Phi_A(\lambda) = (\lambda - \alpha)(\lambda - \beta)$$

with $\alpha, \beta \in \mathbf{R}$ satisfying $\underline{\alpha \neq \beta}$. Moreover we assume that the two vectors $\vec{p}_1, \vec{p}_2$ satisfy

$$A\vec{p}_1 = \alpha\vec{p}_1, \quad A\vec{p}_2 = \beta\vec{p}_2$$

with $\vec{p}_1 \neq \vec{0}$, $\vec{p}_2 \neq \vec{0}$. Then

$$P = (\vec{p}_1 \ \vec{p}_2)$$

is regular.

Proof

Remark that

$$P \text{ is regular} \quad \Leftrightarrow \quad \left(c_1\vec{p}_1 + c_2\vec{p}_2 = \vec{0} \ \Rightarrow c_1 = c_2 = 0 \right).$$

We assume that

$$c_1\vec{p}_1 + c_2\vec{p}_2 = \vec{0}$$

and multiply the both hand sides by $(\beta I_2 - A)$ to get

$$c_1(\beta - \alpha)\vec{p}_1 = \vec{0}$$

It follwos from $\beta - \alpha \neq 0$ and $\vec{p}_1 \neq \vec{0}$ that

$$c_1 = 0$$

In this situation we get

$$c_2\vec{p}_2 = \vec{0}$$

It follows from $\vec{p}_2 \neq \vec{0}$ that

$$c_2 = 0.$$

Example (1)

Let $A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$. Then

$$\Phi_A(\lambda) = \begin{vmatrix} \lambda - 2 & -2 \\ -2 & \lambda - 5 \end{vmatrix} = (\lambda - 1)(\lambda - 6)$$

Thus the eigenvalues of A are $\lambda = 1, 6$. We find the eigenvectors of A as follows.

<u>In case $\lambda = 1$</u>

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \quad \Leftrightarrow \quad x + 2y = 0$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

are the eigenvectors of A for the eigenvalue $\lambda = 1$.

Example (2)

In case $\lambda = 6$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 6\begin{pmatrix} x \\ y \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \quad \Leftrightarrow \quad 2x - y = 0.$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (x \neq 0)$$

are the eigenvectors of A for the eigenvalue $\lambda = 6$.

Example (3)

We choose tow vectors $\vec{q}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \vec{q}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and also a matrix

$$Q = (\vec{q}_1 \ \vec{q}_2) = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

The Q is regular by the Theorem proven above and

$$\begin{aligned} AQ &= (A\vec{q}_1 \ A\vec{q}_2) = (\vec{q}_1 \ 6\vec{q}_2) \\ &= (\vec{q}_1 \ \vec{q}_2) \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} = Q \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \end{aligned}$$

Thus we can diagonalize A by

$$Q^{-1}AQ = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

Symmetric matrices

Remark that

$$\vec{q}_1 \perp \vec{q}_2$$

This is not a coincidence. Actually we have the following theorem.

Theorem

Let A be a 2×2 matirx and assume that A is symmetric *i.e.* ${}^tA = A$. Moreover we assume that

$$A\vec{p} = \alpha\vec{p}, \quad A\vec{q} = \beta\vec{q}$$

with $\alpha, \beta \in \mathbf{R}$ satisfying $\alpha \neq \beta$. Then

$$\vec{p} \perp \vec{q}$$

proof

$$(A\vec{p}, \vec{q}) = (\vec{p}, {}^t A\vec{q}) = (\vec{p}, A\vec{q})$$

Moreover

$$(A\vec{p}, \vec{q}) = (\alpha\vec{p}, \vec{q}) = \alpha(\vec{p}, \vec{q})$$

$$(\vec{p}, A\vec{q}) = (\vec{p}, \beta\vec{q}) = \beta(\vec{p}, \vec{q})$$

These two identities lead us to

$$(\alpha - \beta)(\vec{p}, \vec{q}) = 0$$

Why do we need symmetric matrices?

We are given a symmetric matrix $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$. We introduce the quadratice form for A by

$$\left(A \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) = ax^2 + 2cxy + by^2$$

In case $A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$, we have

$$\left(A \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) = 2x^2 + 4xy + 5y^2$$

We need a more detailed analysis about the diagonalization of symmetric matrices for its application.

Rotation Matrices(1)

We define the rotation matrix of the angle θ by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The rotation can be figured out by the identity

$$R_\theta \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} \cos(\alpha+\theta) \\ \sin(\alpha+\theta) \end{pmatrix}$$

Rotation matrices

Moreover we find that

$$R_\theta^{-1} = \frac{1}{|R_\theta|}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = {}^tR_\theta$$

Accordingly

$${}^tR_\theta \cdot R_\theta = R_\theta \cdot {}^tR_\theta = I_2$$

This identity leads us to

$$(R_\theta\vec{v}, R_\theta\vec{w}) = (\vec{v}, \vec{w})$$

for any $\vec{v}, \vec{w} \in \mathbf{R}^2$. In fact

$$LHS = (\vec{v}, {}^tR_\theta R_\theta \vec{w}) = (\vec{v}, I_2\vec{w}) = (\vec{v}, \vec{w})$$

Diagonalization of symmetric matrices

Let us go back to the example $A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$. We can diagonalize A by a rotation matrix chosen as folloes. We define two unit vectors by

$$\vec{r}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \vec{r}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Then $R = (\vec{r}_1 \ \vec{r}_2)$ is a rotation matrix and we have

$$A\vec{r}_1 = \vec{r}_1, \quad A\vec{r}_2 = 6\vec{r}_2$$

Accordingly

$$AR = (A\vec{r}_1 \ A\vec{r}_2) = (\vec{r}_1 \ 6\vec{r}_2) = (\vec{r}_1 \ \vec{r}_2) \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} = R \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

Thus we have deduced $R^{-1}AR = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$

Diagonalization of symmetric matrices

We now consider the quadratic form of A. Remark that R^{-1} is a rotation. It follwos from this fact that

$$\left(A\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\right)=\left(R^{-1}A\begin{pmatrix}x\\y\end{pmatrix},R^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$$
$$=\left(R^{-1}AR\cdot R^{-1}\begin{pmatrix}x\\y\end{pmatrix},R^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$$
$$=\left(\begin{pmatrix}1&0\\0&6\end{pmatrix}\begin{pmatrix}\xi\\\eta\end{pmatrix},\begin{pmatrix}\xi\\\eta\end{pmatrix}\right)=\xi^2+6\eta^2$$

Here we used a rotaional coordinate transformation defined by

$$\begin{pmatrix}\xi\\\eta\end{pmatrix}=R^{-1}\begin{pmatrix}x\\y\end{pmatrix}$$