Eigenvalue Problems

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December 20, 2016

Review

Theorem

For $B \in M_2(\mathbf{R})$, the following conditions are equivalent:

- **(i)** B is regular.
- **(ii)** $B\vec{v} = \vec{0} \Rightarrow \vec{v} = \vec{0}$
- **(iii)** $|B| \neq 0$

Theorem

For $B \in M_2(\mathbf{R})$, the following conditions are equivalent:

- **NOT(i)** B is not regular (singular).
- **NOT(ii)** There exists $\vec{v} \neq \vec{0}$ satisfying $B\vec{v} = \vec{0}$.
- **NOT(iii)** $|B| = 0$

Example (1)

Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. We consider the system of equations:

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

First remark that the system is equivalent to

$$(\#) \quad (\lambda I_2 - A) \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}, \quad \text{where} \quad \lambda I_2 - A = \begin{pmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{pmatrix}$$

We also remark that

$$|\lambda I_2 - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{vmatrix} = (\lambda + 1)(\lambda - 5)$$

Example (2)

In case $\lambda \neq -1, 5$ $(\#) \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$

In case $\lambda = -1$

$$(\#) \Leftrightarrow \begin{pmatrix} -2 & -2 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Leftrightarrow x + y = 0$$

We put $y = t$ to get the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Example (3)

In case $\lambda = 5$

$$(\#) \Leftrightarrow \begin{pmatrix} 4 & -2 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Leftrightarrow 2x - y = 0$$

We put $y = t$ to get the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ 2t \end{pmatrix} = t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Diagonalization (1)

We define two vectors

$$\vec{p}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \vec{p}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and a 2×2 matrix

$$P = (\vec{p}_1 \ \vec{p}_2) = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$$

The matrix P is regular since

$$|P| = \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = -3 \neq 0$$

Diagonalization (2)

$$\begin{aligned} AP &= A(\vec{p}_1\ \vec{p}_2) = (A\vec{p}_1\ A\vec{p}_2) \\ &= (-\vec{p}_1\ 5\vec{p}_2) \\ &= (\vec{p}_1\ \vec{p}_2)\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} = P\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \end{aligned}$$

We multiply P^{-1} from the left to get

$$P^{-1}AP = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

This process is called a diagonalization of A. To see how it dows mean, we introduce the coordinate transform given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \xi\vec{p}_1 + \eta\vec{p}_2 = (\vec{p}_1\ \vec{p}_2)\begin{pmatrix} \xi \\ \eta \end{pmatrix} = P\begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

Diagonalization (3)

In this situation the map

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

can be clarified by using the coordinate transform as follows.

$$\begin{aligned} \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} &= P^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} \\ &= P^{-1} A \begin{pmatrix} x \\ y \end{pmatrix} \\ &= P^{-1} A P \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \end{aligned}$$