Criterion for the regularity of matrices (1)

Theorem

The following conditions are equivalent for $A \in M_3(\mathbf{R})$

- **(i)** A is regular.
- **(ii)** $A\vec{v} = \vec{0} \Rightarrow \vec{v} = 0$
- **(iii)** $|A| \neq 0$

(iii)$\Rightarrow$(i) is already shown by using the cofactor matrix $\tilde{A}$ of A. If the condition (i) is satisfied, we have

$$A \cdot A^{-1} = I_3$$

Then it follows from the determinants of the both hand side that

$$|A| \cdot |A^{-1}| = |I_3| = 1$$

Accordingly we find $|A| \neq 0$.

Criterion for the regularity of matrices(2)

(i)$\Rightarrow$(ii) We assume that that the condition (i) is satisfied. Then

$$A\vec{v} = \vec{0} \ \rightarrow \ A^{-1}A\vec{v} = A^{-1}\vec{0} = \vec{0} \ \rightarrow \vec{v} = \vec{0}$$

(iii)$\Rightarrow$(ii) We assume that the condition (iii) is satisfied. Then we can apply the Cramer's Rule as follows.

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0} \ \rightarrow \ x = \frac{1}{|A|} \cdot |\vec{0} \ \vec{a}_2 \ \vec{a}_3| = 0 \text{ etc.}$$

Criterion for the regularity of matrices(3)

(ii)⇒(iii) We prove the contraposition **NOT(iii)⇒NOT(ii)**. We assume that $|A| = 0$. Let $A = (\vec{a}\ \vec{b}\ \vec{c})$.
(a) In case $a_1 \neq 0$ We get

$0 \neq$

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \to A_1 \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & b_2' & c_2' \\ 0 & b_3' & c_3' \end{pmatrix} \qquad \begin{matrix} R_2 \to \left(-\frac{a_2}{a_1}\right) R_1 + R_2, \\ R_3 \to \left(-\frac{a_3}{a_1}\right) R_1 + R_3 \end{matrix}$$

$=$

In this situation

$$0 = |A| = |A_1| = a_1 \begin{vmatrix} b_2' & c_2' \\ b_3' & c_3' \end{vmatrix}$$

Then it follwos from $a_1 \neq 0$ that

$\neq 0$

$$\begin{vmatrix} b_2' & c_2' \\ b_3' & c_3' \end{vmatrix} = 0$$

Criterion for the regularity of matrices(4)

Moreover we have the equivalence

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0} \Leftrightarrow \left\{ \begin{array}{rrrcl} a_1 x + & b_1 y + & c_1 z & = & 0 \cdots (1) \\ & b_2' y + & c_2' z & = & 0 \cdots (2)' \\ & b_3' y + & c_3' z & = & 0 \cdots (3)' \end{array} \right.$$

Thanks to the condition $\begin{vmatrix} b_2' & c_2' \\ b_3' & c_3' \end{vmatrix} = 0$ we can find $(y, z) \neq (0, 0)$ satisfying $(2)'$ and $(3)'$. Now it suffices to put

$$x = -\frac{1}{a_1}(b_1 y + c_1 z)$$

Criterion for the regularity of matrices(5)

(b) In case $a_1 = 0,\ a_2 \neq 0$ We get

$\neq 0$

$$A = \begin{pmatrix} 0 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \to A_2 = \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{pmatrix} \qquad R_1 \leftrightarrow R_2$$

In this situation we have the equivalence

$$A\vec{v} = \vec{0} \Leftrightarrow A_2\vec{v} = \vec{0}$$

Moreover

$$a_2 \neq 0, \text{and} |A_2| = -|A| = 0$$

This makes it possible to apply the case (a) to find $\vec{v} \neq \vec{0}$ satisfying $A_2\vec{v} = \vec{0}$. $\rightarrow A\vec{v} = \vec{0}$

Criterion for the regularity of matrices(6)

(c) In case $a_1 = 0,\ a_2 = 0,\ a_3 \neq 0$ We get

$$A = \begin{pmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \to A_3 = \begin{pmatrix} a_3 & b_3 & c_3 \\ 0 & b_2 & c_2 \\ 0 & b_1 & c_1 \end{pmatrix} \qquad R_1 \leftrightarrow R_3$$

In this situation we have the equivalence

$$A\vec{v} = \vec{0} \Leftrightarrow A_3\vec{v} = \vec{0}$$

Moreover

$$a_3 \neq 0, \text{and} |A_3| = -|A| = 0$$

This makes it possible to apply the case (a) to find $\vec{v} \neq \vec{0}$ satisfying $A_3\vec{v} = \vec{0}$. $\rightarrow A\vec{v} = \vec{0}.$

Criterion for the regularity of matrices(7)

(d) In case $a_1 = 0,\ a_2 = 0,\ a_3 = 0$ It suffices to remark that

$$A\begin{pmatrix}1\\0\\0\end{pmatrix} = (\vec{0}\ \vec{b}\ \vec{c})\begin{pmatrix}1\\0\\0\end{pmatrix} = 1\cdot\vec{0} + 0\cdot\vec{b} + 0\cdot\vec{0} = \vec{0}$$

$\neq \vec{0}$

c

Theorem Let $A \in M_n(\mathbb{R})$

$n \times n$ matrix.

(i) A is regular

(ii) $A\vec{v} = \vec{0} \Rightarrow \vec{v} = \vec{0}$

(iii) $|A| \neq 0$.

Gram-Schmidt Orthogonalization

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December 20, 2016

Problem

We are given two vectors

$$\vec{p}, \vec{q} \in \mathbf{R}^n \quad \text{satisfying} \quad \vec{p} \not\parallel \vec{q}$$

We define a subset in $\mathbf{R}^n$ by

$$V := \{x\vec{p} + y\vec{q} \in \mathbf{R}^n;\ x, y \in \mathbf{R}\}$$

~~spanned~~ spaned

called the linear subspace sppaned by $\vec{p}$ and $\vec{q}$. We are given another vector $\vec{c} \in \mathbf{R}^n$. The problem is to find $\vec{c}_0 \in V$ satisfying

$$\vec{c} - \vec{c}_0 \perp V \quad \text{i.e.} \quad (\vec{c} - \vec{c}_0, \vec{v}) = 0 \quad (\vec{v} \in V)$$

$\vec{p}, \vec{q} \in V$

Remark that this condition is equivalent to

$$(\vec{c} - \vec{c}_0, \vec{p}) = (\vec{c} - \vec{c}_0, \vec{q}) = 0$$

$\forall \vec{v} \in V \quad \vec{v} = \xi \vec{p} + \eta \vec{q}$

$(\vec{c} - \vec{c}_0, \vec{v}) = (\vec{c} - \vec{c}_0, \xi\vec{p} + \eta\vec{q})$

$= \xi(\vec{c} - \vec{c}_0, \vec{p}) + \eta(\vec{c} - \vec{c}_0, \vec{q})$

$= 0$

Orthnormal Basis (1)

We take the orthogonal projection of $\vec{q}$ to the direction of $\vec{q}$:

$$\vec{w} = \frac{(\vec{p},\vec{q})}{||\vec{p}||^2}\cdot\vec{p}$$

In this situation we have

$$\vec{q}-\vec{w}\perp\vec{p}$$

Moreover

$$\vec{q}-\vec{w}\neq \vec{0}$$

In fact if $\vec{q}=\vec{w}=*\vec{p}$, then $\vec{p}\parallel\vec{q}$. This contradicts the hypothesis.
We define two vectors

$\vec{p} \not\parallel \vec{q}$

$$\vec{r}_1 = \frac{1}{||\vec{p}||}\vec{p},\quad \vec{r}_2 = \frac{1}{||\vec{q}-\vec{w}||}(\vec{q}-\vec{w})$$

called an orthonormal basis of V.

Orthnormal Basis (2)

The orthonormal basis $\vec{r}_1$ and $\vec{r}_2$ enjoys the basic property

$$||\vec{r}_1|| = ||\vec{r}_2|| = 1,\quad (\vec{r}_1,\vec{r}_2)=0$$

We can express the vector $\vec{c}_0\in V$ by

$$\vec{c}_0 = \xi\vec{r}_1 + \eta\vec{r}_2$$

Moreover $(\vec{c}-\vec{c}_0,\vec{r}_1) = (\vec{c}-\vec{c}_0,\vec{r}_2) = 0$ implies

$$= (\vec{c},\vec{r}_1) - \xi\underbrace{(\vec{r}_1,\vec{r}_1)}_{=1} - \eta\underbrace{(\vec{r}_1,\vec{r}_2)}_{=0}$$

$$0 = (\vec{c}-\vec{c}_0,\vec{r}_1) = (\vec{c}-\xi\vec{r}_1-\eta\vec{r}_2,\vec{r}_1) = (\vec{c},\vec{r}_1)-\xi$$

$$0 = (\vec{c}-\vec{c}_0,\vec{r}_2) = (\vec{c}-\xi\vec{r}_1-\eta\vec{r}_2,\vec{r}_2) = (\vec{c},\vec{r}_2)-\eta$$

$\xi = (\vec{c},\vec{r}_1)$

$\eta = (\vec{c},\vec{r}_2).$

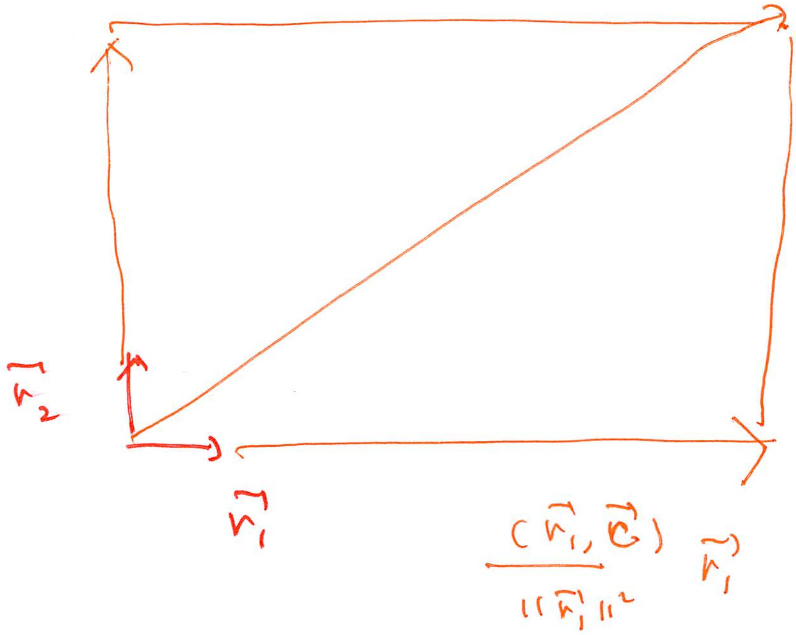
Accordingly we find that

$$\vec{c}_0 = (\vec{c},\vec{r}_1)\vec{r}_1 + (\vec{c},\vec{r}_2)\vec{r}_2$$

$$\vec{c}_0 = (\vec{c}, \vec{r}_1)\,\vec{r}_1 + (\vec{c}, \vec{r}_2)\,\vec{r}_2$$

$\|\vec{r}_1\| = \|\vec{r}_2\| = 1$

$$= \underbrace{\frac{(\vec{c}, \vec{r}_1)}{\|\vec{r}_1\|^2}\,\vec{r}_1}_{\text{Orthogonal projection of } \vec{c} \text{ along } \vec{r}_1} + \underbrace{\frac{(\vec{c}, \vec{r}_2)}{\|\vec{r}_2\|^2}\,\vec{r}_2}_{\text{Orthogonal pr. of } \vec{c} \text{ along } \vec{r}_2}$$



Eigenvalue Problems

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December 20, 2016

Review

Theorem

For $B \in M_2(\mathbf{R})$, the following conditions are equivalent:

- **(i)** B is regular.
- **(ii)** $B\vec{v} = \vec{0} \Rightarrow \vec{v} = \vec{0}$
- **(iii)** $|B| \neq 0$

Theorem

For $B \in M_2(\mathbf{R})$, the following conditions are equivalent:

- **NOT(i)** B is not regular (singular).
- **NOT(ii)** There exists $\vec{v} \neq \vec{0}$ satisfying $B\vec{v} = \vec{0}$.
- **NOT(iii)** $|B| = 0$

Example (1)

Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. We consider the system of equations:

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

First remark that the system is equivalent to

$= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

$$(\#) \quad (\lambda I_2 - A) \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}, \quad \text{where} \quad \lambda I_2 - A = \begin{pmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{pmatrix}$$

We also remark that

$$|\lambda I_2 - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{vmatrix} = (\lambda + 1)(\lambda - 5)$$

$\lambda = -1, 5$ are the Eigen value of A.

Example (2)

In case $\lambda \neq -1, 5$ $(\#) \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$

In case $\lambda = -1$

$-2x - 2y = 0$

$$(\#) \Leftrightarrow \begin{pmatrix} -2 & -2 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Leftrightarrow x + y = 0$$

$-4x - 4y = 0$

We put $y = t$ to get the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$A \begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}$

Example (3)

In case $\lambda = 5$

$$(\#) \Leftrightarrow \begin{pmatrix} 4 & -2 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Leftrightarrow 2x - y = 0$$

We put $y = t$ to get the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ 2t \end{pmatrix} = t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$A \begin{pmatrix} x \\ y \end{pmatrix} = 5 \begin{pmatrix} x \\ y \end{pmatrix}$

Diagonalization (1)

$A\vec{p_1} = -\vec{p_1}$

We define two vectors

$$\vec{p}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \vec{p}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$A\vec{P_2} = 5\vec{P_2}$

and a 2×2 matrix

$$P = (\vec{p}_1 \ \vec{p}_2) = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$$

The matrix P is regular since

$$|P| = \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = -3 \neq 0$$

Diagonalization (2)

$$AP = A(\vec{p}_1\ \vec{p}_2) = (A\vec{p}_1\ A\vec{p}_2)$$
$$= (-\vec{p}_1\ 5\vec{p}_2)$$
$$= (\vec{p}_1\ \vec{p}_2)\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} = P\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

We multiply P^{-1} from the left to get

$$= P^{-1}P\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} = I_2\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

This process is called a diagonalization of A. To see how it dows mean, we introduce the coordinate transform given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \xi\vec{p}_1 + \eta\vec{p}_2 = (\vec{p}_1\ \vec{p}_2)\begin{pmatrix} \xi \\ \eta \end{pmatrix} = P\begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = P^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$ the inverse transform

Diagonalization (3)

$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

In this situation the map

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 4x + 3y \end{pmatrix}$$

can be clarified by using the coordinate transform as follows.

$$\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = P^{-1}\begin{pmatrix} x' \\ y' \end{pmatrix}$$
$$= P^{-1}A\begin{pmatrix} x \\ y \end{pmatrix}$$
$$= P^{-1}AP\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -\xi \\ 5\eta \end{pmatrix}$$

I $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

Find $\lambda \in \mathbb{R}$ for which

$\lambda I_3 - A$ is not regular

II. $\vec{p} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\vec{q} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$

Find an orthonormal basis of

V spanned by $\vec{p}$ and $\vec{q}$.

III Diagonalize $A = \begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix}$