Cofactor Matrix

Let $A \in M_3(\mathbf{R})$ be a 3×3 matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then A_{ij} denotes the 2×2 matrix obtained by deleting ith row and jth column. For example

$$A_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}, \quad A_{23} = \begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix}.$$

Moreover we dfine the (i, j) cofactor of A by

$$\tilde{A}_{ij} = (-1)^{i+j} \det(A_{ij})$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Cofactor Matrix(2)

By using cofactors the matrix $A = (\vec{a}\ \vec{b}\ \vec{c})$ can be expressed in the follwoing way.

$$\begin{aligned} & \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ = & \ a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ = & \ a_1(-1)^{1+1} \det(A_{11}) + a_2(-1)^{2+1} \det(A_{21}) + a_3(-1)^{3+1} \det(A_{31}) \\ = & \ a_{11}\tilde{A}_{11} + a_{21}\tilde{A}_{21} + a_{31}\tilde{A}_{31} \\ = & \ (\tilde{A}_{11}\ \tilde{A}_{21}\ \tilde{A}_{31}) \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \end{aligned}$$

Cofactor Matrix(3)

Moreover the cofactor expansions with respect to the 2nd and the 3rd columns are written in the following way.

$$
\begin{aligned}
|A| &= a_{12}\tilde{A}_{12} + a_{22}\tilde{A}_{22} + a_{32}\tilde{A}_{32} \\
&= (\tilde{A}_{12}\ \tilde{A}_{22}\ \tilde{A}_{32}) \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
|A| &= a_{13}\tilde{A}_{13} + a_{23}\tilde{A}_{23} + a_{33}\tilde{A}_{33} \\
&= (\tilde{A}_{13}\ \tilde{A}_{23}\ \tilde{A}_{33}) \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}
\end{aligned}
$$

$$\tilde{A}\begin{pmatrix} \; \\ \; \\ \; \end{pmatrix} (\vec{a}_1\ \vec{a}_2\ \vec{a}_3) = \begin{pmatrix} |A| & & \\ & |A| & \\ 0 & & |A| \end{pmatrix}$$

Cofactor Matrix(4)

Cofactor Matrix

The cofactor matrix of A is defined by

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{21} & \tilde{A}_{31} \\ \tilde{A}_{12} & \tilde{A}_{22} & \tilde{A}_{32} \\ \tilde{A}_{13} & \tilde{A}_{23} & \tilde{A}_{33} \end{pmatrix}$$

at the 1st row
at the 2nd colum

We calculate $\tilde{A} \cdot A$:

$$\tilde{A} \cdot A = \begin{pmatrix} |A| & *_{12} & *_{13} \\ *_{21} & |A| & *_{23} \\ *_{31} & *_{32} & |A| \end{pmatrix}$$

$*_{12}$

at 2nd row
3rd colum

$*_{23}$

Cofactor Matrix (5)

What happens for $*_{ij}$? For example

$(\vec{a_1} \ \vec{a_2} \ \vec{a_2})$

$$0 = \begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{32} \end{vmatrix}$$

$\vec{a_2}$

$$= \boxed{\begin{pmatrix} \tilde{A}_{13} & \tilde{A}_{23} & \tilde{A}_{33} \end{pmatrix}} \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = *_{32}$$

3rd row of $\tilde{A}$

Accordingly we have shown that

$$\tilde{A} \cdot A = |A| \cdot I_3$$

$$= \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix} = |A| \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Cofactor Expansions for rows(1)

$$\begin{vmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= -b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Cofactor Expansions for rows (2)

We prove the expansion by the 2nd row in the following way.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

take transposition

$$= -b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Cofactor Matrix (6)

The cofactor expansions of three rows turn out the following for $A = (a_{ij})$.

1st column of $\tilde{A}$

$$|A| = a_{11}\tilde{A}_{11} + a_{12}\tilde{A}_{12} + a_{13}\tilde{A}_{13} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix} \begin{pmatrix} \tilde{A}_{11} \\ \tilde{A}_{12} \\ \tilde{A}_{13} \end{pmatrix}$$

2nd c. of $\tilde{A}$

$$|A| = a_{21}\tilde{A}_{21} + a_{22}\tilde{A}_{22} + a_{23}\tilde{A}_{23} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} \tilde{A}_{21} \\ \tilde{A}_{22} \\ \tilde{A}_{23} \end{pmatrix}$$

3rd c of $\tilde{A}$

$$|A| = a_{31}\tilde{A}_{31} + a_{32}\tilde{A}_{32} + a_{33}\tilde{A}_{33} = \begin{pmatrix} a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \tilde{A}_{31} \\ \tilde{A}_{32} \\ \tilde{A}_{33} \end{pmatrix}$$

Cofactor Matrix (7)

Moreover we have the identity for the cofactor matrix $\tilde{A}$ of A.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{21} & \tilde{A}_{31} \\ \tilde{A}_{12} & \tilde{A}_{22} & \tilde{A}_{32} \\ \tilde{A}_{13} & \tilde{A}_{23} & \tilde{A}_{33} \end{pmatrix} = \begin{pmatrix} |A| & \#_{12} & \#_{13} \\ \#_{21} & |A| & \#_{23} \\ \#_{31} & \#_{32} & |A| \end{pmatrix}$$

$= 0$

We can show that $\#_{ij} = 0$ by considering for example

$$0 = \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \cdots$$

$a_{31}\tilde{A}_{11} + a_{32}\tilde{A}_{12} + a_{33}\tilde{A}_{13}$

$\begin{pmatrix} a_{13} \\ a_{12} \\ a_{13} \end{pmatrix}$

$= (a_{31}\ a_{32}\ a_{33}) \begin{pmatrix} \tilde{A}_{11} \\ \tilde{A}_{12} \\ \tilde{A}_{13} \end{pmatrix}$

Cofactor Matrix (8)

Theorem

For $A = (a_{ij}) \in M_3(\mathbf{R})$ we have the identities

$$\tilde{A} \cdot A = A \cdot \tilde{A} = |A| \cdot I_3$$

$= \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}$

Moreover if $|A| \neq 0$, then A is regular and

$$A^{-1} = \frac{1}{|A|}\tilde{A}$$

$\frac{1}{|A|}$ $\quad \frac{1}{|A|}\tilde{A} \cdot A = A \cdot \frac{1}{|A|}\tilde{A} = I_3$

If A is regular, the $|A| \neq 0$.

Another Basic Property for Determinants(1)

Theorem

For $A, X \in M_3(\mathbf{R})$, we have the identity

$$|AX| = |A| \cdot |X|$$

We consider the case where

$$A = (\vec{a}_1\ \vec{a}_2\ \vec{a}_3), \quad X = (\vec{x}\ \vec{y}\ \vec{z})$$

Then

$$AX = (*_1\ \ *_2\ \ *_3)$$

where

$$*_1 = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3,\ *_2 = y_1\vec{a}_1 + y_2\vec{a}_2 + y_3\vec{a}_3,$$
$$*_3 = z_1\vec{a}_1 + z_2\vec{a}_2 + z_3\vec{a}_3$$

$y_1\vec{a}_1 + y_2\vec{a}_2 + y_3\vec{a}_3$

Another Basic Property for Determinants(2)

$*_3 = z_1\vec{a}_1 + z_2\vec{a}_2 + z_3\vec{a}_3$

$|AX|$

$$\begin{aligned}
|A| &= x_1|\vec{a}_1\ \ *_2\ \ *_3| + x_2|\vec{a}_2\ \ *_2\ \ *_3| + x_3|\vec{a}_3\ \ *_2\ \ *_3| \\
&= x_1|\vec{a}_1\ \ y_2\vec{a}_2 + y_3\vec{a}_3\ \ *_3\,| + x_2|\vec{a}_2\ \ y_1\vec{a}_1 + y_3\vec{a}_3\ \ *_3\,| \\
&\quad + x_3|\vec{a}_3\ \ y_1\vec{a}_1 + y_2\vec{a}_2\ \ *_3\,| \\
&= x_1y_2|\vec{a}_1\ \vec{a}_2\ *_3\,| + x_1y_3|\vec{a}_1\ \vec{a}_3\ \ *_3\,| \\
&\quad + x_2y_1|\vec{a}_2\ \vec{a}_1\ \ *_3\,| + x_2y_3|\vec{a}_2\ \vec{a}_3\ \ *_3\,| \\
&\qquad + x_3y_1|\vec{a}_3\ \vec{a}_1\ \ *_3\,| + x_3y_2|\vec{a}_3\ \vec{a}_2\ \ *_3\,| \\
&= x_1y_2z_3|\vec{a}_1\ \ \vec{a}_2\vec{a}_3| + x_1y_3z_2|\vec{a}_1\ \vec{a}_3\ \vec{a}_2| \\
&\quad + x_2y_1z_3|\vec{a}_2\ \vec{a}_1\ \vec{a}_3| + x_2y_3z_1|\vec{a}_2\ \vec{a}_3\ \vec{a}_1| \\
&\qquad + x_3y_1z_2|\vec{a}_3\ \vec{a}_1\ \vec{a}_2| + x_3y_2z_1|\vec{a}_3\ \vec{a}_2\ \vec{a}_1|
\end{aligned}$$

Moreover for any permutation (i, j, k) of $\{1, 2, 3\}$ we have

$$|\vec{a}_i\ \vec{a}_j\ \vec{a}_k| = \varepsilon(i, j, k)|\vec{a}_1\ \vec{a}_2\ \vec{a}_3|$$

Another Basic Property for Determinants(3)

$$\begin{aligned}|A| &= x_1y_2z_3\cdot\varepsilon(1,2,3)|\vec{a}_1\ \vec{a}_2\ \vec{a}_3| + x_1y_3z_2\cdot\varepsilon(1,3,2)|\vec{a}_1\ \vec{a}_2\ \vec{a}_3|\\ &\quad + x_2y_1z_3\cdot\varepsilon(2,1,3)|\vec{a}_1\ \vec{a}_2\ \vec{a}_3| + x_2y_3z_1\cdot\varepsilon(2,3,1)|\vec{a}_1\ \vec{a}_2\ \vec{a}_3|\\ &\quad + x_3y_1z_2\cdot\varepsilon(3,1,2)|\vec{a}_1\ \vec{a}_2\ \vec{a}_3| + x_3y_2z_1\cdot\varepsilon(3,2,1)|\vec{a}_1\ \vec{a}_2\ \vec{a}_3|\\ &= \sum_{(i,j,k)}\varepsilon(i,j,k)x_iy_jz_k\cdot|\vec{a}_1\ \vec{a}_2\ \vec{a}_3|\\ &= |X|\cdot|A|\end{aligned}$$

If A is regular

$A\cdot A^{-1} = I_3$

Take det(·)

$|A\cdot A^{-1}| = |I_3| = 1$ $= |A|\cdot|A^{-1}|$

Criterion for the regularity of matrices

Theorem

The following conditions are equivalent for $A \in M_3(\mathbf{R})$

- **(i)** A is regular.
- **(ii)** $A\vec{v} = \vec{0} \Rightarrow \vec{v} = 0$
- **(iii)** $|A| \neq 0$

(iii)⇒(i) is already shown by using the cofactor matrix $\tilde{A}$ of A. If the condition (i) is satisfied, we have

$$A\cdot A^{-1} = I_3$$

Then it follows from the determinants of the both hand side that

$$|A|\cdot|A^{-1}| = |I_3| = 1$$

Accordingly we find $|A| \neq 0$.