Exercises for TA Session 2016/12/06

I Find the value of the determinants:

(1) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ (2) $\begin{vmatrix} 2 & -1 & -3 \\ 1 & 1 & 1 \\ 4 & 5 & 6 \end{vmatrix}$

(3) $\begin{vmatrix} 2 & 4 & 6 \\ 0 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$ (4) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

II Assume $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ satisfy

$$\vec{a} = \lambda \vec{b} + \mu \vec{c}.$$

Then find

$$|\vec{a}\ \vec{b}\ \vec{c}| = 0$$

III Solve the systems of equations by the Cramer's Rule.

(1) $\begin{pmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

(2) $\begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

IV Given two vectors

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

We define

$$\vec{b} \times \vec{c} = \begin{pmatrix} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \\ - \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \\ \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \end{pmatrix}$$

(1) Show that

$$(\vec{b}, \vec{b} \times \vec{c}) = (\vec{c}, \vec{b} \times \vec{c}) = 0$$

by applying a Basic Property of 3×3 matrices.

(2) Show that

$$\vec{b} \times \vec{c} = - \vec{c} \times \vec{b}$$

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I (1)

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix} = 1 \begin{vmatrix} -3 & -6 \\ -6 & -12 \end{vmatrix} = 1 \times 0 = 0.$$

(2)
$$\begin{vmatrix} 2 & -1 & -3 \\ 1 & 1 & 1 \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 0 & -3 & -5 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = - \begin{vmatrix} -3 & -5 \\ 1 & 2 \end{vmatrix} = -(-6+5)$$

$$= 1$$

(3)
$$\begin{vmatrix} 2 & 4 & 6 \\ 0 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 0 & -4 & -12 \\ 0 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} -4 & -12 \\ 2 & 3 \end{vmatrix}$$

$$= -4 \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = -12.$$

(4)
$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$= \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$$= (b-a)(c-a)(c-b)$$

$$= (a-b)(b-c)(c-a)$$

II $|\vec{a}\ \vec{b}\ \vec{c}| = |\lambda\vec{b}+\mu\vec{c}\ \ \vec{b}\ \ \vec{c}|$

$= \lambda\,|\vec{b}\ \vec{b}\ \vec{c}| + \mu\,|\vec{c}\ \vec{b}\ \vec{c}|$

$= \lambda\cdot 0 + \mu\cdot 0 = 0$

III (1) $D = \begin{vmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2 \\ 0 & 5 & 0 \\ -1 & 1 & -1 \end{vmatrix}$

$= 5 \begin{vmatrix} 0 & 2 \\ -1 & -1 \end{vmatrix} = 10 \neq 0.$

$x = \frac{1}{D}\begin{vmatrix} 1 & -2 & 4 \\ 2 & 3 & 2 \\ 3 & 1 & -1 \end{vmatrix} = \frac{1}{10}\begin{vmatrix} 1 & -2 & 4 \\ 0 & 7 & -6 \\ 0 & 7 & -13 \end{vmatrix}$

$= \frac{1}{10}\begin{vmatrix} 7 & -6 \\ 7 & -13 \end{vmatrix} = \frac{1}{10}\begin{vmatrix} 7 & -6 \\ 0 & -7 \end{vmatrix} = -\frac{49}{10}$

$y = \frac{1}{D}\begin{vmatrix} 2 & 1 & 4 \\ 2 & 2 & 2 \\ -1 & 3 & -1 \end{vmatrix} = \frac{1}{10}\begin{vmatrix} 0 & 7 & 2 \\ 0 & 8 & 0 \\ -1 & 3 & -1 \end{vmatrix} = -\frac{1}{10}\begin{vmatrix} 7 & 2 \\ 8 & 0 \end{vmatrix}$

$= \frac{16}{10} = \frac{8}{5}$

$z = \frac{1}{D}\begin{vmatrix} 2 & -2 & 1 \\ 2 & 3 & 2 \\ -1 & 1 & 3 \end{vmatrix} = \frac{1}{10}\begin{vmatrix} 0 & 0 & 6 \\ 0 & 5 & 8 \\ -1 & 1 & 3 \end{vmatrix} = -\frac{1}{10}\begin{vmatrix} 0 & 6 \\ 5 & 8 \end{vmatrix}$

$= -\frac{1}{10}(-30) = 3$

$\boxed{\text{II}}$ (2)

$$\begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 2 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & -3 \end{vmatrix} = 2 \begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix}$$

$$= 2 \, (-3+2) = -2 \neq 0 .$$

$$x = \frac{1}{D} \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & -1 \\ 1 & 3 & -1 \end{vmatrix} = -\frac{1}{2} \begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & -5 \\ 0 & 2 & -3 \end{vmatrix}$$

$$= -\frac{1}{2} \begin{vmatrix} -1 & -5 \\ 2 & -3 \end{vmatrix} = -\frac{1}{2} \, (3+10) = -\frac{13}{2}$$

$$y = \frac{1}{D} \begin{vmatrix} 2 & 1 & 2 \\ 0 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} = -\frac{1}{2} \begin{vmatrix} 2 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & -3 \end{vmatrix} = -\frac{1}{2} \left(2 \times 2 \times (-3) \right)$$

$$= 6$$

$$z = \frac{1}{D} \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = -\frac{1}{2} \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{vmatrix}$$

$$= -\frac{1}{2} \times (-2) \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = 4$$

IV (1)

$$(\vec{b}, \vec{b}\times\vec{c}) = b_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = 0$$

$$(\vec{c}, \vec{b}\times\vec{c}) = c_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + c_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} = 0$$

(2)

$$\vec{b}\times\vec{c} = \begin{pmatrix} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \\ -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \\ \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -\begin{vmatrix} c_2 & b_2 \\ c_3 & b_3 \end{vmatrix} \\ \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \\ -\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \end{pmatrix}$$

$$= -\begin{pmatrix} \begin{vmatrix} c_2 & b_2 \\ c_3 & b_3 \end{vmatrix} \\ -\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \\ \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \end{pmatrix} = -\vec{c}\times\vec{b}$$