Determinants for 3×3 Matrices

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Definition

We are given three 3-dimensional column vectors

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \ \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \ \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

and a 3×3 matrix

$$A = (\vec{a}\ \vec{b}\ \vec{c})$$

We define the determinant of A by

$$\begin{aligned} \det(A) = |A| &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \end{aligned}$$

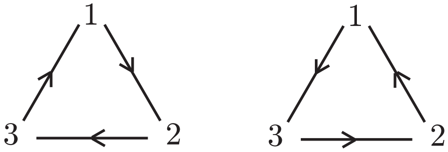
Sarrus Rule

We expand the three determinants of 2×2 matrices to get

$$|A| = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1$$

The signs in the head of $a_i b_j c_k$ are determined by the following rule.

$$\varepsilon(i,j,k) = \begin{cases} 1 & ((i,j,k) \text{ has positive orientation}) \\ -1 & ((i,j,k) \text{ has negative orientation}) \end{cases}$$



Positive Orientation Negative Orientation

Sarrus Rule(2)

$$\det(A) = \sum_{i\neq j,\ j\neq k, k\neq i} \varepsilon(i\ j\ k) \cdot a_i b_j c_k \tag{1}$$

Cofactor Expansion

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$= c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Basic Property I

Basic Property I – Multi-linearity

$$\left|(s_1\vec{\alpha}+s_2\vec{\beta})\ \vec{b}\ \vec{c}\right| = s_1\left|\vec{\alpha}\ \vec{b}\ \vec{c}\right| + s_2\left|\vec{\beta}\ \vec{b}\ \vec{c}\right|$$

$$\left|\vec{a}\ (s_1\vec{\alpha}+s_2\vec{\beta})\ \vec{c}\right| = s_1\left|\vec{a}\ \vec{\alpha}\ \vec{c}\right| + s_2\left|\vec{a}\ \vec{\beta}\ \vec{c}\right|$$

$$\left|\vec{a}\ \vec{b}\ (s_1\vec{\alpha}+s_2\vec{\beta})\right| = s_1\left|\vec{a}\ \vec{b}\ \vec{\alpha}\right| + s_2\left|\vec{a}\ \vec{b}\ \vec{\beta}\right|$$

Basic Property I

The basic Property I is based on the following theorem.

Theorem

$F: \ \mathbf{R}^3 \to \mathbf{R}$ is defined by

$$F(\vec{x}) = b_1x_1 + b_2x_2 + b_3x_3$$

Then we have

$$F(c_1\vec{\alpha} + c_2\vec{\beta}) = c_1F(\vec{\alpha}) + c_3F(\vec{\beta})$$

Basic Properties II+III

Basic Property II – Interchanging two colums

$$|\vec{a}\ \vec{b}\ \vec{c}| = -|\vec{b}\ \vec{a}\ \vec{c}| = -|\vec{a}\ \vec{c}\ \vec{b}| = -|\vec{c}\ \vec{b}\ \vec{a}|$$

$$\begin{aligned} |\vec{a}\ \vec{b}\ \vec{c}| &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ &= -a_1 \begin{vmatrix} c_2 & b_2 \\ c_3 & b_3 \end{vmatrix} + a_2 \begin{vmatrix} c_1 & b_1 \\ c_3 & b_3 \end{vmatrix} - a_3 \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \\ &= -|\vec{a}\ \vec{c}\ \vec{b}| \end{aligned}$$

Basic Property III

$$|I_3| = 1$$

Basic Property IV

Basic Property III

$$|A| = |{}^tA| \quad i.e. \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

To prove this, develop the RHS:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \cdots$$

Basic Property IR

Basic Property IR – Multi-linearity

$$\begin{vmatrix} \mathbf{a} \\ s_1\mathbf{p} + s_2\mathbf{q} \\ \mathbf{c} \end{vmatrix} = s_1 \begin{vmatrix} \mathbf{a} \\ \mathbf{p} \\ \mathbf{c} \end{vmatrix} + s_2 \begin{vmatrix} \mathbf{a} \\ \mathbf{q} \\ \mathbf{c} \end{vmatrix}$$

$$\begin{aligned} LHS &= |{}^t\mathbf{a}\ {}^t(s_1\mathbf{p} + s_2\mathbf{q})\ {}^t\mathbf{c}| \\ &= |{}^t\mathbf{a}\ s_1{}^t\mathbf{p} + s_2{}^t\mathbf{q}\ {}^t\mathbf{c}| \\ &= s_1|{}^t\mathbf{a}\ {}^t\mathbf{p}\ {}^t\mathbf{c}| + s_2|{}^t\mathbf{a}\ {}^t\mathbf{q}\ {}^t\mathbf{c}| = RHS \end{aligned}$$

Basic Property IIR

Basic Property IIR–Interchanging two rows

$$\begin{vmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{vmatrix} = -\begin{vmatrix} \mathbf{b} \\ \mathbf{a} \\ \mathbf{c} \end{vmatrix} = -\begin{vmatrix} \mathbf{a} \\ \mathbf{c} \\ \mathbf{b} \end{vmatrix} = -\begin{vmatrix} \mathbf{c} \\ \mathbf{b} \\ \mathbf{a} \end{vmatrix}$$

Basic Properties V+VR and VIR

Basic Properties V+VR

If A has two idential colums (rows):

$$|\vec{a}\ \vec{a}\ \vec{c}| = 0, \qquad \begin{vmatrix} \mathbf{a} \\ \mathbf{a} \\ \mathbf{c} \end{vmatrix} = 0$$

Basic Property VIR

Adding a multiple of a row to another:

$$\begin{vmatrix} \mathbf{a} \\ \mu\mathbf{a} + \mathbf{b} \\ \mathbf{c} \end{vmatrix} = \begin{vmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{vmatrix} \tag{2}$$

example

$$\begin{vmatrix} 0 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 4 & 9 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & 9 \\ 2 & 4 & 6 \\ 0 & 2 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & 9 \\ 0 & -4 & -12 \\ 0 & 2 & 3 \end{vmatrix}$$

$$= -\begin{vmatrix} -4 & -12 \\ 2 & 3 \end{vmatrix} = -(-4)\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 4(3-6) = -12$$

Cramer's Rule

We study the following system of equations:

$$\begin{cases} a_1x + b_1y + c_1z &= \alpha_1 & \cdots (1) \\ a_2x + b_2y + c_2z &= \alpha_2 & \cdots (2) \\ a_3x + b_3y + c_3z &= \alpha_3 & \cdots (3) \end{cases}$$

Cramer's Rule (2)

To elimeinate z, we consider $(1) \times c_2 - (2) \times c_1$:

$$\begin{array}{rccll}
 & a_1c_2x + b_1c_2y + c_1c_2z & = & \alpha_1c_2 & \cdots(1)\times c_2 \\
-) & a_2c_1x + b_2c_1y + c_1c_2z & = & \alpha_2c_1 & \cdots(2)\times c_1 \\
\hline
 & \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} x + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} y & = & \begin{vmatrix} \alpha_1 & c_1 \\ \alpha_2 & c_2 \end{vmatrix} & \cdots(\mathrm{I})
\end{array}$$

We also consider $(1) \times c_3 - (3) \times c_1$ and $(2) \times c_3 - (3) \times c_2$to get

$$\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} x + \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} y = \begin{vmatrix} \alpha_1 & c_1 \\ \alpha_3 & c_3 \end{vmatrix} \quad \cdots(\mathrm{II}) = (1)\times c_3 - (3)\times c_1$$

$$\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} x + \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} y = \begin{vmatrix} \alpha_2 & c_2 \\ \alpha_3 & c_3 \end{vmatrix} \quad \cdots(\mathrm{III}) = (2)\times c_3 - (3)\times c_2$$

Cramer's Rule (3)

We consider $-b_1 \times (III) + b_2 \times (II) - b_3 \times (I)$ to get

$$|\vec{a}\ \vec{b}\ \vec{c}|x + |\vec{b}\ \vec{b}\ \vec{c}|y = |\vec{\alpha}\ \vec{b}\ \vec{c}|$$

It follows from $|\vec{b}\ \vec{b}\ \vec{c}| = 0$ that

$$|\vec{a}\ \vec{b}\ \vec{c}|x = |\vec{\alpha}\ \vec{b}\ \vec{c}|$$

Accordingly if $D := |\vec{a}\ \vec{b}\ \vec{c}| \neq 0$, we find

$$x = \frac{1}{D}|\vec{\alpha}\ \vec{b}\ \vec{c}|$$

In the same way as above, we can derive

$$y = \frac{1}{D}|\vec{a}\ \vec{\alpha}\ \vec{c}|, \quad z = \frac{1}{D}|\vec{a}\ \vec{b}\ \vec{\alpha}|$$

Cofactor Matrix

Let $A \in M_3(\mathbf{R})$ be a 3×3 matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then A_{ij} denotes the 2×2 matrix obtained by deleting ith row and jth column. For example

$$A_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}, \ A_{23} = \begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix}.$$

Moreover we dfine the (i, j) cofactor of A by

$$\tilde{A}_{ij} = (-1)^{i+j} \det(A_{ij})$$

Cofactor Matrix(2)

By using cofactors the matrix $A = (\vec{a}\ \vec{b}\ \vec{c})$ can be expressed in the follwoing way.

$$
\begin{aligned}
&\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\
= \;& a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\
= \;& a_1(-1)^{1+1}\det(A_{11}) + a_2(-1)^{2+1}\det(A_{21}) + a_3(-1)^{3+1}\det(A_{31}) \\
= \;& a_{11}\tilde{A}_{11} + a_{21}\tilde{A}_{21} + a_{31}\tilde{A}_{31} \\
= \;& (\tilde{A}_{11}\ \tilde{A}_{21}\ \tilde{A}_{31}) \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}
\end{aligned}
$$

Cofactor Matrix(3)

Moreover the cofactor expansions with respect to the 2nd and the 3rd columns are written in the following way.

$$
\begin{aligned}
|A| &= a_{12}\tilde{A}_{12} + a_{22}\tilde{A}_{22} + a_{32}\tilde{A}_{32} \\
&= (\tilde{A}_{12}\ \tilde{A}_{22}\ \tilde{A}_{32}) \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
|A| &= a_{13}\tilde{A}_{13} + a_{23}\tilde{A}_{23} + a_{33}\tilde{A}_{33} \\
&= (\tilde{A}_{13}\ \tilde{A}_{23}\ \tilde{A}_{33}) \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}
\end{aligned}
$$

Cofactor Matrix(4)

Cofactor Matrix

The cofactor matrix of A is defined by

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{21} & \tilde{A}_{31} \\ \tilde{A}_{12} & \tilde{A}_{22} & \tilde{A}_{32} \\ \tilde{A}_{13} & \tilde{A}_{23} & \tilde{A}_{33} \end{pmatrix}$$

We calculate $\tilde{A} \cdot A$:

$$\tilde{A} \cdot A = \begin{pmatrix} |A| & *_{12} & *_{13} \\ *_{21} & |A| & *_{23} \\ *_{31} & *_{32} & |A| \end{pmatrix}$$

Cofactor Matrix (5)

What happens for $*_{ij}$? For example

$$0 = \begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{32} \end{vmatrix}$$

$$= \left(\tilde{A}_{13} \; \tilde{A}_{23} \; \tilde{A}_{33} \right) \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = *_{32}$$

Accordingly we have shown that

$$\tilde{A} \cdot A = |A| \cdot I_3$$

Cofactor Expansions for rows(1)

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= -b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Cofactor Expansions for rows (2)

We prove the expansion by the 2nd row in the following way.

$$\begin{aligned}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \\ &= -b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}\end{aligned}$$

Cofactor Matrix (6)

The cofactor expansions of three rows turn out the following for $A = (a_{ij})$.

$$|A| = a_{11}\tilde{A}_{11} + a_{12}\tilde{A}_{12} + a_{13}\tilde{A}_{13} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix} \begin{pmatrix} \tilde{A}_{11} \\ \tilde{A}_{12} \\ \tilde{A}_{13} \end{pmatrix}$$

$$|A| = a_{21}\tilde{A}_{21} + a_{22}\tilde{A}_{22} + a_{23}\tilde{A}_{23} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} \tilde{A}_{21} \\ \tilde{A}_{22} \\ \tilde{A}_{23} \end{pmatrix}$$

$$|A| = a_{31}\tilde{A}_{31} + a_{32}\tilde{A}_{32} + a_{33}\tilde{A}_{33} = \begin{pmatrix} a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \tilde{A}_{31} \\ \tilde{A}_{32} \\ \tilde{A}_{33} \end{pmatrix}$$

Cofactor Matrix (7)

Moreover we have the identity for the cofactor matrix $\tilde{A}$ of A.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{21} & \tilde{A}_{31} \\ \tilde{A}_{12} & \tilde{A}_{22} & \tilde{A}_{32} \\ \tilde{A}_{13} & \tilde{A}_{23} & \tilde{A}_{33} \end{pmatrix} = \begin{pmatrix} |A| & \#_{12} & \#_{13} \\ \#_{21} & |A| & \#_{23} \\ \#_{31} & \#_{32} & |A| \end{pmatrix}$$

We can show that $\#_{ij} = 0$ by considering for example

$$0 = \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \cdots$$

Cofactor Matrix (8)

Theorem

For $A = (a_{ij}) \in M_3(\mathbf{R})$ we have the identities

$$\tilde{A} \cdot A = A \cdot \tilde{A} = |A| \cdot I_3$$

Moreover if $|A| \neq 0$, then A is regular and

$$A^{-1} = \frac{1}{|A|}\tilde{A}$$

Another Basic Property for Determinants(1)

Theorem

For $A, X \in M_3(\mathbf{R})$, we have the identity

$$|AX| = |A| \cdot |X|$$

We consider the case where

$$A = (\vec{a}_1 \;\; \vec{a}_2 \;\; \vec{a}_3), \quad X = (\vec{x} \;\; \vec{y} \;\; \vec{z})$$

Then

$$AX = (*_1 \;\; *_2 \;\; *_3)$$

where

$$*_1 = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3, \; *_2 = y_1\vec{a}_1 + y_2\vec{a}_2 + y_3\vec{a}_3,$$
$$*_3 = z_1\vec{a}_1 + z_2\vec{a}_2 + z_3\vec{a}_3$$

Another Basic Property for Determinants(2)

$$
\begin{aligned}
|A| &= x_1|\vec{a}_1 \ \ *_2 \ \ *_3| + x_2|\vec{a}_2 \ \ *_2 \ \ *_3| + x_3|\vec{a}_3 \ \ *_2 \ \ *_3| \\
&= x_1|\vec{a}_1 \ y_2\vec{a}_2 + y_3\vec{a}_3 \ \ *_3 \ | + x_2|\vec{a}_2 \ y_1\vec{a}_1 + y_3\vec{a}_3 \ \ *_3 \ | \\
&\qquad + x_3|\vec{a}_3 \ y_1\vec{a}_1 + y_2\vec{a}_2 \ \ *_3 \ | \\
&= x_1y_2|\vec{a}_1 \ \vec{a}_2 *_3 \ | + x_1y_3|\vec{a}_1 \ \vec{a}_3 \ \ *_3 \ | \\
&\qquad + x_2y_1|\vec{a}_2 \ \vec{a}_1 \ \ *_3 \ | + x_2y_3|\vec{a}_2 \ \vec{a}_3 \ \ *_3 \ | \\
&\qquad\qquad + x_3y_1|\vec{a}_3 \ \vec{a}_1 \ \ *_3 \ | + x_3y_2|\vec{a}_3 \ \vec{a}_2 \ \ *_3 \ | \\
&= x_1y_2z_3|\vec{a}_1 \ \vec{a}_2\vec{a}_3| + x_1y_3z_2|\vec{a}_1 \ \vec{a}_3 \ \vec{a}_2| \\
&\qquad + x_2y_1z_3|\vec{a}_2 \ \vec{a}_1 \ \vec{a}_3| + x_2y_3z_1|\vec{a}_2 \ \vec{a}_3 \ \vec{a}_1| \\
&\qquad\qquad + x_3y_1z_2|\vec{a}_3 \ \vec{a}_1 \ \vec{a}_2| + x_3y_2z_1|\vec{a}_3 \ \vec{a}_2 \ \vec{a}_1|
\end{aligned}
$$

Moreover for any permutation (i, j, k) of $\{1, 2, 3\}$ we have

$$
|\vec{a}_i \ \vec{a}_j \ \vec{a}_k| = \varepsilon(i, j, k)|\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3|
$$

Another Basic Property for Determinants(3)

$$
\begin{aligned}
|AX| &= x_1y_2z_3\cdot\varepsilon(1,2,3)|\vec{a}_1\ \vec{a}_2\ \vec{a}_3| + x_1y_3z_2\cdot\varepsilon(1,3,2)|\vec{a}_1\ \vec{a}_2\ \vec{a}_3| \\
&\quad + x_2y_1z_3\cdot\varepsilon(2,1,3)|\vec{a}_1\ \vec{a}_2\ \vec{a}_3| + x_2y_3z_1\cdot\varepsilon(2,3,1)|\vec{a}_1\ \vec{a}_2\ \vec{a}_3| \\
&\quad + x_3y_1z_2\cdot\varepsilon(3,1,2)|\vec{a}_1\ \vec{a}_2\ \vec{a}_3| + x_3y_2z_1\cdot\varepsilon(3,2,1)|\vec{a}_1\ \vec{a}_2\ \vec{a}_3| \\
&= \sum_{(i,j,k)} \varepsilon(i,j,k)x_iy_jz_k\cdot|\vec{a}_1\ \vec{a}_2\ \vec{a}_3| \\
&= |X|\cdot|A|
\end{aligned}
$$

Criterion for the regularity of matrices (1)

Theorem

The following conditions are equivalent for $A \in M_3(\mathbf{R})$

- **(i)** A is regular.
- **(ii)** $A\vec{v} = \vec{0} \Rightarrow \vec{v} = \vec{0}$
- **(iii)** $|A| \neq 0$

(iii)$\Rightarrow$(i) is already shown by using the cofactor matrix $\tilde{A}$ of A. If the condition (i) is satisfied, we have

$$A \cdot A^{-1} = I_3$$

Then it follows from the determinants of the both hand side that

$$|A| \cdot |A^{-1}| = |I_3| = 1$$

Accordingly we find $|A| \neq 0$.

Criterion for the regularity of matrices(2)

<u>**(i)⇒(ii)**</u> We assume that that the condition (i) is satisfied. Then

$$A\vec{v} = \vec{0} \ \rightarrow \ A^{-1}A\vec{v} = A^{-1}\vec{0} = \vec{0} \ \rightarrow \vec{v} = \vec{0}$$

<u>**(iii)⇒(ii)**</u> We assume that the condition (iii) is satisfied. Then we can apply the Cramer's Rule as follows.

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0} \ \rightarrow \ x = \frac{1}{|A|} \cdot |\vec{0} \ \vec{a}_2 \ \vec{a}_3| = 0 \text{ etc.}$$

Criterion for the regularity of matrices(3)

(ii)⇒(iii) We prove the contraposition **NOT(iii)⇒NOT(ii)**. We assume that $|A| = 0$. Let $A = (\vec{a}\ \vec{b}\ \vec{c})$.

(a) In case $a_1 \neq 0$ We get

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \to A_1 = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & b_2' & c_2' \\ 0 & b_3' & c_3' \end{pmatrix} \qquad \begin{matrix} R_2 \to \left(-\frac{a_2}{a_1}\right) R_1 + R_2, \\ R_3 \to \left(-\frac{a_3}{a_1}\right) R_1 + R_3 \end{matrix}$$

In this situation

$$0 = |A| = |A_1| = a_1 \begin{vmatrix} b_2' & c_2' \\ b_3' & c_3' \end{vmatrix}$$

Then it follwos from $a_1 \neq 0$ that

$$\begin{vmatrix} b_2' & c_2' \\ b_3' & c_3' \end{vmatrix} = 0$$

Criterion for the regularity of matrices(4)

Moreover we have the equivalence

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0} \Leftrightarrow \left\{ \begin{array}{cccc} a_1x+ & b_1y+ & c_1z & = & 0\cdots(1) \\ & b_2'y+ & c_2'z & = & 0\cdots(2)' \\ & b_3'y+ & c_3'z & = & 0\cdots(3)' \end{array} \right.$$

Thanks to the condition $\begin{vmatrix} b_2' & c_2' \\ b_3' & c_3' \end{vmatrix} = 0$ we can find $(y, z) \neq (0, 0)$ satisfying $(2)'$ and $(3)'$. Now it suffices to put

$$x = -\frac{1}{a_1}(b_1y + c_1z)$$

(b) In case $a_1 = 0,\ a_2 \neq 0$ We get

$$A = \begin{pmatrix} 0 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \to A_2 = \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{pmatrix} \quad R_1 \leftrightarrow R_2$$

In this situation we have the equivalence

$$A\vec{v} = \vec{0} \Leftrightarrow A_2\vec{v} = \vec{0}$$

Moreover

$$a_2 \neq 0, \text{and} |A_2| = -|A| = 0$$

This makes it possible to apply the case (a) to find $\vec{v} \neq \vec{0}$ satisfying $A_2\vec{v} = \vec{0}$.

Criterion for the regularity of matrices(6)

(c) In case $a_1 = 0,\ a_2 = 0,\ \ a_3 \neq 0$ We get

$$A = \begin{pmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \to A_3 = \begin{pmatrix} a_3 & b_3 & c_3 \\ 0 & b_2 & c_2 \\ 0 & b_1 & c_1 \end{pmatrix} \quad R_1 \leftrightarrow R_3$$

In this situation we have the equivalence

$$A\vec{v} = \vec{0} \Leftrightarrow A_3\vec{v} = \vec{0}$$

Moreover

$$a_3 \neq 0, \text{and} |A_3| = -|A| = 0$$

This makes it possible to apply the case (a) to find $\vec{v} \neq \vec{0}$ satisfying $A_3\vec{v} = \vec{0}$.

Criterion for the regularity of matrices(7)

(d) In case $a_1 = 0,\ a_2 = 0,\ \ a_3 = 0$ It suffices to remark that

$$A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (\vec{0}\ \vec{b}\ \vec{c}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 \cdot \vec{0} + 0 \cdot \vec{b} + 0 \cdot \vec{0} = \vec{0}$$