

Determinants for 3×3 Matrices

Nobuyuki TOSE

December 06, 2016

Definition

We are given three 3-dimensional column vectors

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

and a 3×3 matrix

$$A = (\vec{a} \ \vec{b} \ \vec{c})$$

We define the determinant of A by

$$\begin{aligned} \det(A) = |A| &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \end{aligned}$$

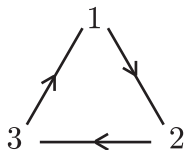
Sarrus Rule

We expand the three determinants of 2×2 matrices to get

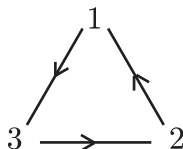
$$|A| = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1$$

The signs in the head of $a_i b_j c_k$ are determined by the following rule.

$$\varepsilon(i, j, k) = \begin{cases} 1 & ((i, j, k) \text{ has positive orientation}) \\ -1 & ((i, j, k) \text{ has negative orientation}) \end{cases}$$



Positive Orientation



Negative Orientation

Sarrus Rule(2)

$$\det(A) = \sum_{i \neq j, j \neq k, k \neq i} \varepsilon(i \ j \ k) \cdot a_i b_j c_k \quad (1)$$

Cofactor Expansion

$$\begin{aligned} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ &= -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \\ &= c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{aligned}$$

Basic Property I

Basic Property I – Multi-linearity

$$\left| (s_1 \vec{\alpha} + s_2 \vec{\beta}) \vec{b} \vec{c} \right| = s_1 \left| \vec{\alpha} \vec{b} \vec{c} \right| + s_2 \left| \vec{\beta} \vec{b} \vec{c} \right|$$

$$\left| \vec{a} (s_1 \vec{\alpha} + s_2 \vec{\beta}) \vec{c} \right| = s_1 \left| \vec{a} \vec{\alpha} \vec{c} \right| + s_2 \left| \vec{a} \vec{\beta} \vec{c} \right|$$

$$\left| \vec{a} \vec{b} (s_1 \vec{\alpha} + s_2 \vec{\beta}) \right| = s_1 \left| \vec{a} \vec{b} \vec{\alpha} \right| + s_2 \left| \vec{a} \vec{b} \vec{\beta} \right|$$

Basic Property I

The basic Property I is based on the following theorem.

Theorem

$F : \mathbf{R}^3 \rightarrow \mathbf{R}$ is defined by

$$F(\vec{x}) = b_1x_1 + b_2x_2 + b_3x_3$$

Then we have

$$F(c_1\vec{\alpha} + c_2\vec{\beta}) = c_1F(\vec{\alpha}) + c_2F(\vec{\beta})$$

Basic Properties II+III

Basic Property II – Interchanging two columns

$$|\vec{a} \ \vec{b} \ \vec{c}| = -|\vec{b} \ \vec{a} \ \vec{c}| = -|\vec{a} \ \vec{c} \ \vec{b}| = -|\vec{c} \ \vec{b} \ \vec{a}|$$

$$\begin{aligned} |\vec{a} \ \vec{b} \ \vec{c}| &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ &= -a_1 \begin{vmatrix} c_2 & b_2 \\ c_3 & b_3 \end{vmatrix} + a_2 \begin{vmatrix} c_1 & b_1 \\ c_3 & b_3 \end{vmatrix} - a_3 \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \\ &= -|\vec{a} \ \vec{c} \ \vec{b}| \end{aligned}$$

Basic Property III

$$|I_3| = 1$$

Basic Property IV

Basic Property III

$$|A| = |{}^t A| \quad \text{i.e.} \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

To prove this, develop the RHS:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \dots$$

Basic Property IR

Basic Property IR – Multi-linearity

$$\begin{vmatrix} \mathbf{a} \\ s_1\mathbf{p} + s_2\mathbf{q} \\ \mathbf{c} \end{vmatrix} = s_1 \begin{vmatrix} \mathbf{a} \\ \mathbf{p} \\ \mathbf{c} \end{vmatrix} + s_2 \begin{vmatrix} \mathbf{a} \\ \mathbf{q} \\ \mathbf{c} \end{vmatrix}$$

$$\begin{aligned} LHS &= |{}^t\mathbf{a} \ {}^t(s_1\mathbf{p} + s_2\mathbf{q}) \ {}^t\mathbf{c}| \\ &= |{}^t\mathbf{a} \ s_1{}^t\mathbf{p} + s_2{}^t\mathbf{q} \ {}^t\mathbf{c}| \\ &= s_1|{}^t\mathbf{a} \ {}^t\mathbf{p} \ {}^t\mathbf{c}| + s_2|{}^t\mathbf{a} \ {}^t\mathbf{q} \ {}^t\mathbf{c}| = RHS \end{aligned}$$

Basic Property IIR

Basic Property IIR–Interchanging two rows

$$\begin{vmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{vmatrix} = - \begin{vmatrix} \mathbf{b} \\ \mathbf{a} \\ \mathbf{c} \end{vmatrix} = - \begin{vmatrix} \mathbf{a} \\ \mathbf{c} \\ \mathbf{b} \end{vmatrix} = - \begin{vmatrix} \mathbf{c} \\ \mathbf{b} \\ \mathbf{a} \end{vmatrix}$$

Basic Properties V+VR and VIR

Basic Properties V+VR

If A has two identical columns (rows):

$$|\vec{a} \ \vec{a} \ \vec{c}| = 0, \quad \begin{vmatrix} \mathbf{a} \\ \mathbf{a} \\ \mathbf{c} \end{vmatrix} = 0$$

Basic Property VIR

Adding a multiple of a row to another:

$$\begin{vmatrix} \mathbf{a} \\ \mu\mathbf{a} + \mathbf{b} \\ \mathbf{c} \end{vmatrix} = \begin{vmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{vmatrix} \quad (2)$$

example

$$\begin{vmatrix} 0 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 4 & 9 \end{vmatrix} = - \begin{vmatrix} 1 & 4 & 9 \\ 2 & 4 & 6 \\ 0 & 2 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 4 & 9 \\ 0 & -4 & -12 \\ 0 & 2 & 3 \end{vmatrix} \\ = - \begin{vmatrix} -4 & -12 \\ 2 & 3 \end{vmatrix} = -(-4) \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 4(3 - 6) = -12$$

Cramer's Rule

We study the following system of equations:

$$\begin{cases} a_1x + b_1y + c_1z = \alpha_1 & \cdots (1) \\ a_2x + b_2y + c_2z = \alpha_2 & \cdots (2) \\ a_3x + b_3y + c_3z = \alpha_3 & \cdots (3) \end{cases}$$

Cramer's Rule (2)

To eliminate z , we consider $(1) \times c_2 - (2) \times c_1$:

$$\begin{array}{rcl} a_1 c_2 x + b_1 c_2 y + c_1 c_2 z & = & \alpha_1 c_2 \quad \cdots (1) \times c_2 \\ -) & & \\ a_2 c_1 x + b_2 c_1 y + c_1 c_2 z & = & \alpha_2 c_1 \quad \cdots (2) \times c_1 \\ \hline \left| \begin{array}{cc} a_1 & c_1 \\ a_2 & c_2 \end{array} \right| x + \left| \begin{array}{cc} b_1 & c_1 \\ b_2 & c_2 \end{array} \right| y & = & \left| \begin{array}{cc} \alpha_1 & c_1 \\ \alpha_2 & c_2 \end{array} \right| \quad \cdots (I) \end{array}$$

We also consider $(1) \times c_3 - (3) \times c_1$ and $(2) \times c_3 - (3) \times c_2$ to get

$$\left| \begin{array}{cc} a_1 & c_1 \\ a_3 & c_3 \end{array} \right| x + \left| \begin{array}{cc} b_1 & c_1 \\ b_3 & c_3 \end{array} \right| y = \left| \begin{array}{cc} \alpha_1 & c_1 \\ \alpha_3 & c_3 \end{array} \right| \quad \cdots (II) = (1) \times c_3 - (3) \times c_1$$

$$\left| \begin{array}{cc} a_2 & c_2 \\ a_3 & c_3 \end{array} \right| x + \left| \begin{array}{cc} b_2 & c_2 \\ b_3 & c_3 \end{array} \right| y = \left| \begin{array}{cc} \alpha_2 & c_2 \\ \alpha_3 & c_3 \end{array} \right| \quad \cdots (III) = (2) \times c_3 - (3) \times c_2$$

Cramer's Rule (3)

We consider $-b_1 \times (III) + b_2 \times (II) - b_3 \times (I)$ to get

$$|\vec{a} \ \vec{b} \ \vec{c}|x + |\vec{b} \ \vec{b} \ \vec{c}|y = |\vec{\alpha} \ \vec{b} \ \vec{c}|$$

It follows from $|\vec{b} \ \vec{b} \ \vec{c}| = 0$ that

$$|\vec{a} \ \vec{b} \ \vec{c}|x = |\vec{\alpha} \ \vec{b} \ \vec{c}|$$

Accordingly if $D := |\vec{a} \ \vec{b} \ \vec{c}| \neq 0$, we find

$$x = \frac{1}{D} |\vec{\alpha} \ \vec{b} \ \vec{c}|$$

In the same way as above, we can derive

$$y = \frac{1}{D} |\vec{a} \ \vec{\alpha} \ \vec{c}|, \quad z = \frac{1}{D} |\vec{a} \ \vec{b} \ \vec{\alpha}|$$

Cofactor Matrix

Let $A \in M_3(\mathbf{R})$ be a 3×3 matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then A_{ij} denotes the 2×2 matrix obtained by deleting i th row and j th column. For example

$$A_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}, \quad A_{23} = \begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix}.$$

Moreover we define the (i, j) cofactor of A by

$$\tilde{A}_{ij} = (-1)^{i+j} \det(A_{ij})$$

Cofactor Matrix(2)

By using cofactors the matrix $A = (\vec{a} \ \vec{b} \ \vec{c})$ can be expressed in the following way.

$$\begin{aligned} & \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ &= a_1(-1)^{1+1} \det(A_{11}) + a_2(-1)^{2+1} \det(A_{21}) + a_3(-1)^{3+1} \det(A_{31}) \\ &= a_{11}\tilde{A}_{11} + a_{21}\tilde{A}_{21} + a_{31}\tilde{A}_{31} \\ &= (\tilde{A}_{11} \ \tilde{A}_{21} \ \tilde{A}_{31}) \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \end{aligned}$$

Cofactor Matrix(3)

Moreover the cofactor expansions with respect to the 2nd and the 3rd columns are written in the following way.

$$\begin{aligned}|A| &= a_{12}\tilde{A}_{12} + a_{22}\tilde{A}_{22} + a_{32}\tilde{A}_{32} \\ &= (\tilde{A}_{12} \ \tilde{A}_{22} \ \tilde{A}_{32}) \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}\end{aligned}$$

$$\begin{aligned}|A| &= a_{13}\tilde{A}_{13} + a_{23}\tilde{A}_{23} + a_{33}\tilde{A}_{33} \\ &= (\tilde{A}_{13} \ \tilde{A}_{23} \ \tilde{A}_{33}) \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}\end{aligned}$$

Cofactor Matrix(4)

Cofactor Matrix

The cofactor matrix of A is defined by

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{21} & \tilde{A}_{31} \\ \tilde{A}_{12} & \tilde{A}_{22} & \tilde{A}_{32} \\ \tilde{A}_{13} & \tilde{A}_{23} & \tilde{A}_{33} \end{pmatrix}$$

We calculate $\tilde{A} \cdot A$:

$$\tilde{A} \cdot A = \begin{pmatrix} |A| & *_{12} & *_{13} \\ *_{21} & |A| & *_{23} \\ *_{31} & *_{32} & |A| \end{pmatrix}$$

Cofactor Matrix (5)

What happens for $*_{ij}$? For example

$$\begin{aligned} 0 &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= \begin{pmatrix} \tilde{A}_{13} & \tilde{A}_{23} & \tilde{A}_{33} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = *_{32} \end{aligned}$$

Accordingly we have shown that

$$\tilde{A} \cdot A = |A| \cdot I_3$$

Cofactor Expansions for rows(1)

$$\begin{aligned}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= -b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \\ &= c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\end{aligned}$$

Cofactor Expansions for rows (2)

We prove the expansion by the 2nd row in the following way.

$$\begin{aligned}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \\ &= -b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}\end{aligned}$$

Cofactor Matrix (6)

The cofactor expansions of three rows turn out the following for $A = (a_{ij})$.

$$|A| = a_{11}\tilde{A}_{11} + a_{12}\tilde{A}_{12} + a_{13}\tilde{A}_{13} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix} \begin{pmatrix} \tilde{A}_{11} \\ \tilde{A}_{12} \\ \tilde{A}_{13} \end{pmatrix}$$

$$|A| = a_{21}\tilde{A}_{21} + a_{22}\tilde{A}_{22} + a_{23}\tilde{A}_{23} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} \tilde{A}_{21} \\ \tilde{A}_{22} \\ \tilde{A}_{23} \end{pmatrix}$$

$$|A| = a_{31}\tilde{A}_{31} + a_{32}\tilde{A}_{32} + a_{33}\tilde{A}_{33} = \begin{pmatrix} a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \tilde{A}_{31} \\ \tilde{A}_{32} \\ \tilde{A}_{33} \end{pmatrix}$$

Cofactor Matrix (7)

Moreover we have the identity for the cofactor matrix $\tilde{A}$ of A .

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{21} & \tilde{A}_{31} \\ \tilde{A}_{12} & \tilde{A}_{22} & \tilde{A}_{32} \\ \tilde{A}_{13} & \tilde{A}_{23} & \tilde{A}_{33} \end{pmatrix} = \begin{pmatrix} |A| & \#_{12} & \#_{13} \\ \#_{21} & |A| & \#_{23} \\ \#_{31} & \#_{32} & |A| \end{pmatrix}$$

We can show that $\#_{ij} = 0$ by considering for example

$$0 = \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \cdots$$

Cofactor Matrix (8)

Theorem

For $A = (a_{ij}) \in M_3(\mathbf{R})$ we have the identities

$$\tilde{A} \cdot A = A \cdot \tilde{A} = |A| \cdot I_3$$

Moreover if $|A| \neq 0$, then A is regular and

$$A^{-1} = \frac{1}{|A|} \tilde{A}$$

Another Basic Property for Determinants(1)

Theorem

For $A, X \in M_3(\mathbf{R})$, we have the identity

$$|AX| = |A| \cdot |X|$$

We consider the case where

$$A = (\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3), \quad X = (\vec{x} \ \vec{y} \ \vec{z})$$

Then

$$AX = (*_1 \ *_2 \ *_3)$$

where

$$*_1 = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3, \quad *_2 = y_1\vec{a}_1 + y_2\vec{a}_2 + y_3\vec{a}_3,$$

$$*_3 = z_1\vec{a}_1 + z_2\vec{a}_2 + z_3\vec{a}_3$$

Another Basic Property for Determinants(2)

$$\begin{aligned} |A| &= x_1 |\vec{a}_1 \ *_2 \ *_3| + x_2 |\vec{a}_2 \ *_2 \ *_3| + x_3 |\vec{a}_3 \ *_2 \ *_3| \\ &= x_1 |\vec{a}_1 \ y_2 \vec{a}_2 + y_3 \vec{a}_3 \ *_3| + x_2 |\vec{a}_2 \ y_1 \vec{a}_1 + y_3 \vec{a}_3 \ *_3| \\ &\quad + x_3 |\vec{a}_3 \ y_1 \vec{a}_1 + y_2 \vec{a}_2 \ *_3| \\ &= x_1 y_2 |\vec{a}_1 \ \vec{a}_2 \ *_3| + x_1 y_3 |\vec{a}_1 \ \vec{a}_3 \ *_3| \\ &\quad + x_2 y_1 |\vec{a}_2 \ \vec{a}_1 \ *_3| + x_2 y_3 |\vec{a}_2 \ \vec{a}_3 \ *_3| \\ &\quad + x_3 y_1 |\vec{a}_3 \ \vec{a}_1 \ *_3| + x_3 y_2 |\vec{a}_3 \ \vec{a}_2 \ *_3| \\ &= x_1 y_2 z_3 |\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3| + x_1 y_3 z_2 |\vec{a}_1 \ \vec{a}_3 \ \vec{a}_2| \\ &\quad + x_2 y_1 z_3 |\vec{a}_2 \ \vec{a}_1 \ \vec{a}_3| + x_2 y_3 z_1 |\vec{a}_2 \ \vec{a}_3 \ \vec{a}_1| \\ &\quad + x_3 y_1 z_2 |\vec{a}_3 \ \vec{a}_1 \ \vec{a}_2| + x_3 y_2 z_1 |\vec{a}_3 \ \vec{a}_2 \ \vec{a}_1| \end{aligned}$$

Moreover for any permutation (i, j, k) of $\{1, 2, 3\}$ we have

$$|\vec{a}_i \ \vec{a}_j \ \vec{a}_k| = \varepsilon(i, j, k) |\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3|$$

Another Basic Property for Determinants(3)

$$\begin{aligned} |AX| &= x_1 y_2 z_3 \cdot \varepsilon(1, 2, 3) |\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3| + x_1 y_3 z_2 \cdot \varepsilon(1, 3, 2) |\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3| \\ &\quad + x_2 y_1 z_3 \cdot \varepsilon(2, 1, 3) |\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3| + x_2 y_3 z_1 \cdot \varepsilon(2, 3, 1) |\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3| \\ &\quad + x_3 y_1 z_2 \cdot \varepsilon(3, 1, 2) |\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3| + x_3 y_2 z_1 \cdot \varepsilon(3, 2, 1) |\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3| \\ &= \sum_{(i,j,k)} \varepsilon(i, j, k) x_i y_j z_k \cdot |\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3| \\ &= |X| \cdot |A| \end{aligned}$$

Criterion for the regularity of matrices

Theorem

The following conditions are equivalent for $A \in M_3(\mathbf{R})$

- **(i)** A is regular.
- **(ii)** $A\vec{v} = \vec{0} \Rightarrow \vec{v} = 0$
- **(iii)** $|A| \neq 0$

(iii) $\Rightarrow$ (i) is already shown by using the cofactor matrix $\tilde{A}$ of A . If the condition (i) is satisfied, we have

$$A \cdot A^{-1} = I_3$$

Then it follows from the determinants of the both hand side that

$$|A| \cdot |A^{-1}| = |I_3| = 1$$

Accordingly we find $|A| \neq 0$.