Determinants for 3×3 Matrices

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Definition

We are given three 3-dimensional column vectors

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \ \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \ \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

and a 3×3 matrix

$$A = (\vec{a}\ \vec{b}\ \vec{c})$$

We define the determinant of A by

$$\begin{aligned} \det(A) = |A| &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \end{aligned}$$

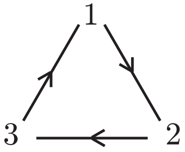
Sarrus Rule

We expand the three determinants of 2×2 matrices to get

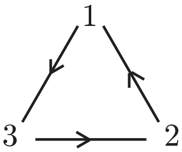
$$|A| = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1$$

The signs in the head of $a_i b_j c_k$ are determined by the following rule.

$$\varepsilon(i,j,k) = \begin{cases} 1 & ((i,j,k) \text{ has positive orientation}) \\ -1 & ((i,j,k) \text{ has negative orientation}) \end{cases}$$



Positive Orientation



Negative Orientation

Sarrus Rule(2)

$$\det(A) = \sum_{i \neq j,\ j \neq k, k \neq i} \varepsilon(i\ j\ k) \cdot a_i b_j c_k \tag{1}$$

Cofactor Expansion

$$
\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}
\begin{aligned}
&= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\
&= -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \\
&= c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}
\end{aligned}
$$

Basic Property I

Basic Property I – Multi-linearity

$$\left|(s_1\vec{\alpha}+s_2\vec{\beta})\ \vec{b}\ \vec{c}\right| = s_1\left|\vec{\alpha}\ \vec{b}\ \vec{c}\right| + s_2\left|\vec{\beta}\ \vec{b}\ \vec{c}\right|$$

$$\left|\vec{a}\ (s_1\vec{\alpha}+s_2\vec{\beta})\ \vec{c}\right| = s_1\left|\vec{a}\ \vec{\alpha}\ \vec{c}\right| + s_2\left|\vec{a}\ \vec{\beta}\ \vec{c}\right|$$

$$\left|\vec{a}\ \vec{b}\ (s_1\vec{\alpha}+s_2\vec{\beta})\right| = s_1\left|\vec{a}\ \vec{b}\ \vec{\alpha}\right| + s_2\left|\vec{a}\ \vec{b}\ \vec{\beta}\right|$$

Basic Property I

The basic Property I is based on the following theorem.

Theorem

$F: \mathbf{R}^3 \to \mathbf{R}$ is defined by

$$F(\vec{x}) = b_1x_1 + b_2x_2 + b_3x_3$$

Then we have

$$F(c_1\vec{\alpha} + c_2\vec{\beta}) = c_1F(\vec{\alpha}) + c_3F(\vec{\beta})$$

Basic Properties II+III

Basic Property II – Interchanging two colums

$$|\vec{a}\ \vec{b}\ \vec{c}| = -|\vec{b}\ \vec{a}\ \vec{c}| = -|\vec{a}\ \vec{c}\ \vec{b}| = -|\vec{c}\ \vec{b}\ \vec{a}|$$

$$\begin{aligned}
|\vec{a}\ \vec{b}\ \vec{c}| &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\
&= -a_1 \begin{vmatrix} c_2 & b_2 \\ c_3 & b_3 \end{vmatrix} + a_2 \begin{vmatrix} c_1 & b_1 \\ c_3 & b_3 \end{vmatrix} - a_3 \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \\
&= -|\vec{a}\ \vec{c}\ \vec{b}|
\end{aligned}$$

Basic Property III

$$|I_3| = 1$$

Basic Property IV

Basic Property III

$$|A| = |{}^tA| \quad i.e. \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

To prove this, develop the RHS:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \cdots$$

Basic Property IR

Basic Property IR – Multi-linearity

$$\begin{vmatrix} \mathbf{a} \\ s_1\mathbf{p} + s_2\mathbf{q} \\ \mathbf{c} \end{vmatrix} = s_1 \begin{vmatrix} \mathbf{a} \\ \mathbf{p} \\ \mathbf{c} \end{vmatrix} + s_2 \begin{vmatrix} \mathbf{a} \\ \mathbf{q} \\ \mathbf{c} \end{vmatrix}$$

$$\begin{aligned} LHS &= |{}^t\mathbf{a}\ {}^t(s_1\mathbf{p} + s_2\mathbf{q})\ {}^t\mathbf{c}| \\ &= |{}^t\mathbf{a}\ s_1{}^t\mathbf{p} + s_2{}^t\mathbf{q}\ {}^t\mathbf{c}| \\ &= s_1|{}^t\mathbf{a}\ {}^t\mathbf{p}\ {}^t\mathbf{c}| + s_2|{}^t\mathbf{a}\ {}^t\mathbf{q}\ {}^t\mathbf{c}| = RHS \end{aligned}$$

Basic Property IIR

Basic Property IIR–Interchanging two rows

$$\begin{vmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{vmatrix} = - \begin{vmatrix} \mathbf{b} \\ \mathbf{a} \\ \mathbf{c} \end{vmatrix} = - \begin{vmatrix} \mathbf{a} \\ \mathbf{c} \\ \mathbf{b} \end{vmatrix} = - \begin{vmatrix} \mathbf{c} \\ \mathbf{b} \\ \mathbf{a} \end{vmatrix}$$

Basic Properties V+VR and VIR

Basic Properties V+VR

If A has two idential colums (rows):

$$|\vec{a}\ \vec{a}\ \vec{c}| = 0, \qquad \begin{vmatrix} \mathbf{a} \\ \mathbf{a} \\ \mathbf{c} \end{vmatrix} = 0$$

Basic Property VIR

Adding a multiple of a row to another:

$$\begin{vmatrix} \mathbf{a} \\ \mu\mathbf{a} + \mathbf{b} \\ \mathbf{c} \end{vmatrix} = \begin{vmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{vmatrix} \tag{2}$$

example

$$\begin{vmatrix} 0 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 4 & 9 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & 9 \\ 2 & 4 & 6 \\ 0 & 2 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & 9 \\ 0 & -4 & -12 \\ 0 & 2 & 3 \end{vmatrix}$$

$$= -\begin{vmatrix} -4 & -12 \\ 2 & 3 \end{vmatrix} = -(-4)\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 4(3-6) = -12$$

Cramer's Rule

We study the following system of equations:

$$\begin{cases} a_1x + b_1y + c_1z &= \alpha_1 & \cdots (1) \\ a_2x + b_2y + c_2z &= \alpha_2 & \cdots (2) \\ a_3x + b_3y + c_3z &= \alpha_3 & \cdots (3) \end{cases}$$

Cramer's Rule (2)

To elimeinate z, we consider $(1) \times c_2 - (2) \times c_1$:

$$
\begin{array}{rrcll}
 & a_1c_2x + b_1c_2y + c_1c_2z & = & \alpha_1c_2 & \cdots(1)\times c_2 \\
-) & a_2c_1x + b_2c_1y + c_1c_2z & = & \alpha_2c_1 & \cdots(2)\times c_1 \\
\hline
 & \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} x + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} y & = & \begin{vmatrix} \alpha_1 & c_1 \\ \alpha_2 & c_2 \end{vmatrix} & \cdots(\mathrm{I})
\end{array}
$$

We also consider $(1) \times c_3 - (3) \times c_1$ and $(2) \times c_3 - (3) \times c_2$to get

$$
\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} x + \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} y = \begin{vmatrix} \alpha_1 & c_1 \\ \alpha_3 & c_3 \end{vmatrix} \quad \cdots(\mathrm{II}) = (1)\times c_3 - (3)\times c_1
$$

$$
\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} x + \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} y = \begin{vmatrix} \alpha_2 & c_2 \\ \alpha_3 & c_3 \end{vmatrix} \quad \cdots(\mathrm{III}) = (2)\times c_3 - (3)\times c_2
$$

Cramer's Rule (3)

We consider $-b_1 \times (III) + b_2 \times (II) - b_3 \times (I)$ to get

$$|\vec{a}\ \vec{b}\ \vec{c}|x + |\vec{b}\ \vec{b}\ \vec{c}|y = |\vec{\alpha}\ \vec{b}\ \vec{c}|$$

It follows from $|\vec{b}\ \vec{b}\ \vec{c}| = 0$ that

$$|\vec{a}\ \vec{b}\ \vec{c}|x = |\vec{\alpha}\ \vec{b}\ \vec{c}|$$

Accordingly if $D := |\vec{a}\ \vec{b}\ \vec{c}| \neq 0$, we find

$$x = \frac{1}{D}|\vec{\alpha}\ \vec{b}\ \vec{c}|$$

In the same way as above, we can derive

$$y = \frac{1}{D}|\vec{a}\ \vec{\alpha}\ \vec{c}|, \quad z = \frac{1}{D}|\vec{a}\ \vec{b}\ \vec{\alpha}|$$