I We take the following into account.

$$\frac{1}{\sqrt{n}}(z_j - \bar{z}) = \frac{1}{\sqrt{n}}\{(a x_j + b y_j + c) - (a\bar{x} + b\bar{y} + c)\}$$

$$= a\frac{1}{\sqrt{n}}(x_j - \bar{x}) + b\frac{1}{\sqrt{n}}(y_j - \bar{y})$$

Then we find that

$$\vec{z} = a\vec{x} + b\vec{y}$$

and it follows that

$$V(z) = \|\vec{z}\|^2 = \|a\vec{x} + b\vec{y}\|^2$$

$$= a\|\vec{x}\|^2 + 2ab(\vec{x}, \vec{y}) + b\|\vec{y}\|^2$$

$$= a V(x) + 2ab C_{xy} + b V(y)$$

II

$$A = \left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 1 & 3 & 2 & 1 \\ 2 & 11 & 5 & 5 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 5 & 1 & 2 \\ 0 & 15 & 3 & 7 \end{array}\right) \quad \begin{array}{l} R_2 \to (-1)R_1 + R_2 \\ R_3 \to (-2)R_1 + R_3 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 5 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right) \quad R_3 \to (-3)R_2 + R_3.$$

It follows from the above sequence of row elementary operation that the given system of equations is equivalent to

$$\begin{cases} x - 2y + z = -1 & \cdots (1)' \\ 5y + z = 2 & \cdots (2)' \\ 0x + 0y + 0z = 1 & \cdots (3)'. \end{cases}$$

Since $(3)'$ has no solution, the given equation has no solution.

III (1)

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 5 & 6 & 3 & 7 & 1 \\ 2 & 1 & 4 & 0 & 6 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 2 & -4 \\ 0 & -1 & 2 & -2 & 4 \end{array}\right) \quad \begin{array}{l} R_2 \to (-5)R_1 + R_2 \\ R_3 \to (\end{array}$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 5 \\ 0 & 1 & -2 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} R_1 \to R_1 + (-1) R_2 \\ R_3 \to R_2 + R_3 \end{array}$$

It follows that the given system of equations is equivalent to

$$\begin{cases} x \quad - w = 5 \\ y - 2z + 2w = -4 \end{cases}$$

Put $z = s$ and $w = t$. Then the solution is expressed by

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} t + 5 \\ 2s - 2t - 4 \\ s \\ t \end{pmatrix}$$

(2)
$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 2 \\ 3 & 2 & 0 & 1 & 1 \\ 4 & 1 & 2 & 2 & 3 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 2 \\ 0 & 5 & -6 & -2 & -5 \\ 0 & 5 & -6 & -2 & -5 \end{array}\right) \quad \begin{array}{l} R_2 \to (-3)R_1 + R_2 \\ R_3 \to (-4)R_1 + R_3 \end{array}$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 2 \\ 0 & 1 & -\frac{6}{5} & -\frac{2}{5} & -1 \\ 0 & 5 & -6 & -2 & -5 \end{array}\right) \quad R_2 \to \frac{1}{5} R_2$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & \frac{4}{5} & \frac{3}{5} & 1 \\ 0 & 1 & -\frac{6}{5} & -\frac{2}{5} & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to (-5)R_2 + R_3 \end{array}$$

It follows that the given system of equations is equivalent to

$$\begin{cases} x + \frac{4}{5} z + \frac{3}{5} w = 1 \\ y - \frac{6}{5} z - \frac{2}{5} w = -1 \end{cases}$$

Put $z = s$ and $w = t$ to express the solution by

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -\frac{4}{5}s - \frac{3}{5}t + 1 \\ \frac{6}{5}s + \frac{2}{5}t - 1 \\ s \\ t \end{pmatrix}$$

(3)
$$\left(\begin{array}{ccc|c} 2 & 6 & 0 & 11 \\ 6 & 20 & -6 & 3 \\ 0 & 6 & -18 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 0 & \frac{11}{2} \\ 6 & 20 & -6 & 3 \\ 0 & 6 & -18 & 1 \end{array}\right) \qquad R_1 \to \frac{1}{2} R_1$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 0 & \frac{11}{2} \\ 0 & 2 & -6 & -30 \\ 0 & 6 & -18 & 1 \end{array}\right) \qquad R_2 \to (-6) R_1 + R_2$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 0 & \frac{11}{2} \\ 0 & 2 & -6 & -30 \\ 0 & 0 & 0 & 91 \end{array}\right) \qquad R_3 \to (-3) R_2 + R_3$$

It follow that the given system of EQS is equivalent to

$$\begin{cases} x + 3y = \frac{11}{2} & \cdots (1)' \\ 2y - 6z = -30 & \cdots (2)' \\ 0x + 0y + 0z = 91 & \cdots (3)' \end{cases}$$

Since $(3)'$ has no solution, the given system has no solution.

IV

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & -3 & 2 & 4 \\ 3 & -2 & 1 & c \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -3 & 2 & 4 \\ 2 & 1 & -1 & 1 \\ 3 & -2 & 1 & c \end{array}\right) \quad R_1 \leftrightarrow R_2$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -3 & 2 & 4 \\ 0 & 7 & -5 & -7 \\ 0 & 7 & -5 & c-12 \end{array}\right) \quad \begin{array}{l} R_2 \to (-2)R_1 + R_2 \\ R_3 \to (-3)R_1 + R_3 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -3 & 2 & 4 \\ 0 & 7 & -5 & -7 \\ 0 & 0 & 0 & c-5 \end{array}\right) \quad R_3 \to (-1)R_2 + R_3.$$

If $c \neq 5$, the system equivalent to the given

$$\begin{cases} x - 3y + 2z = 4 \\ 7y - 5z = -7 \\ 0x + 0y + 0z = c-5 \end{cases}$$

has no solution. If $c = 5$, the last matrix transformed by row elementary operations as follows

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & 4 \\ 0 & 7 & -5 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -3 & 2 & 4 \\ 0 & 1 & -\frac{5}{7} & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & 1 \\ 0 & 1 & -\frac{5}{7} & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus the given system is equivalent to

$$\begin{cases} x - \frac{1}{7} z = 1 \\ y - \frac{5}{7} z = -1 \end{cases}$$

We put $z = t$ to express the solution by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{7}t + 1 \\ \frac{5}{7}t - 1 \\ t \end{pmatrix}.$$

Accordingly if $c = 5$, the system has a(n) (unique) solution. We have shown the condition for the given system has a solution is $c = 5$.

V (1)

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 4 & 1 & 0 & 1 & 0 & 0 \\ 3 & 3 & 3 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -3 & -2 & -3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 \to (-2)R_1 + R_2 \\ R_3 \to (-3)R_1 + R_3 \\ R_4 \to (-1)R_1 + R_4 \end{array}$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 3 & -1 & 0 & 0 \\ 0 & 1 & 2 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -3 & -2 & -3 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 \to R_1 + (-1)R_2 \\ R_4 \to (-1)R_2 + R_4 \end{array}$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 3 & -1 & 0 & 0 \\ 0 & 1 & 2 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & -3 & -2 & -3 & 0 & 1 & 0 \end{array}\right) \quad R_3 \leftrightarrow R_4$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 3 & -1 & 0 & 0 \\ 0 & 1 & 2 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & -1 \\ 0 & 0 & -3 & -2 & -3 & 0 & 1 & 0 \end{array}\right) \quad \begin{array}{l} R_3 \to (-1)R_3 \\ R_4 \to 3R_3 + R_4 \end{array}$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 3 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -5 & -6 & 3 & 1 & -3 \end{array}\right) \quad \begin{array}{l} R_2 \to R_2 + (-2)R_3 \\ R_4 \to 3R_3 + R_4 \end{array}$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 3 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & \frac{6}{5} & -\frac{3}{5} & -\frac{1}{5} & \frac{3}{5} \end{array}\right) \quad R_4 \to -\frac{1}{5} R_4$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3/5 & 1/5 & 2/5 & -6/5 \\ 0 & 1 & 0 & 0 & -6/5 & -2/5 & 1/5 & 7/5 \\ 0 & 0 & 1 & 0 & 1/5 & 2/5 & -1/5 & -2/5 \\ 0 & 0 & 0 & 1 & \frac{6}{5} & -\frac{3}{5} & -\frac{1}{5} & \frac{3}{5} \end{array}\right)$$

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 3 & 1 & 2 & -6 \\ -6 & -2 & 1 & 7 \\ 1 & 2 & -1 & -2 \\ 6 & -3 & -1 & 3 \end{pmatrix}$$

(2)

$$\left(\begin{array}{ccc|ccc} 2 & -2 & 4 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \to \frac{1}{2} R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & \frac{1}{2} & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{R_2 \to (-2) R_1 + R_2 \\ R_3 \to R_1 + R_3}]{} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 5 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 \to \frac{1}{5} R_2]{} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{5} & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 + R_2]{} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{8}{5} & \frac{3}{10} & \frac{1}{5} & 0 \\ 0 & 1 & -\frac{2}{5} & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{R_1 \to R_1 + (-\frac{8}{5}) R_3 \\ R_2 \to R_2 + \frac{2}{5} R_3}]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{5} & -\frac{8}{5} \\ 0 & 1 & 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 1 \end{array}\right)$$

$\therefore$

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{5} & -\frac{8}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$$

(3)

$$\left(\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1 \to (-1) R_1]{} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{R_2 \to (-2) R_1 + R_2 \\ R_3 \to (-4) R_1 + R_3}]{} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to -R_2 + R_3]{} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 & -1 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 1 & -1 & 2 & -1 & 1 \\ 0 & 5 & -6 & 2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{R_1 \to R_1 + 2R_2 \\ R_3 \to (-5) R_2 + R_3}]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -2 & 2 \\ 0 & 1 & -1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -8 & 6 & -5 \end{array}\right)$$

$$\xrightarrow[R_3 \to (-1) R_3]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -2 & 2 \\ 0 & 1 & -1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right)$$

$$\xrightarrow[\substack{R_1 \to R_1 + (-1) R_3 \\ R_2 \to R_2 + R_3}]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 4 & 3 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -5 & 4 & 3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{pmatrix}$$

(4) $\left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 5 & 2 & -3 & 0 & 0 & 1 \end{array}\right)$

$$\xrightarrow[R_3 \to (-2)R_1 + R_3]{} \left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & -5 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1 \leftrightarrow R_3]{} \left(\begin{array}{ccc|ccc} 1 & 2 & -5 & -2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 \to (-2)R_1 + R_3]{} \left(\begin{array}{ccc|ccc} 1 & 2 & -5 & -2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -4 & 11 & 5 & 0 & -2 \end{array}\right)$$

$$\xrightarrow[R_2 \to \frac{1}{2} R_2]{} \left(\begin{array}{ccc|ccc} 1 & 2 & -5 & -2 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -4 & 11 & 5 & 0 & -2 \end{array}\right)$$

$$\xrightarrow[R_3 \to 4R_2 + R_3]{} \left(\begin{array}{ccc|ccc} 1 & 0 & -6 & -2 & -1 & 1 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 13 & 5 & 2 & -2 \end{array}\right)$$

$$\xrightarrow[R_3 \to \frac{1}{13} R_3]{} \left(\begin{array}{ccc|ccc} 1 & 0 & -6 & -2 & -1 & 1 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{5}{13} & \frac{2}{13} & -\frac{2}{13} \end{array}\right)$$

$$\xrightarrow[\substack{R_1 \to R_1 + 6R_3 \\ R_2 \to R_2 + (-\frac{1}{2}) R_3}]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{4}{13} & -\frac{1}{13} & \frac{1}{13} \\ 0 & 1 & 0 & -\frac{5}{26} & \frac{11}{26} & \frac{1}{13} \\ 0 & 0 & 1 & \frac{5}{13} & \frac{2}{13} & -\frac{2}{13} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{4}{13} & -\frac{1}{13} & \frac{1}{13} \\ -\frac{5}{26} & \frac{11}{26} & \frac{1}{13} \\ \frac{5}{13} & \frac{2}{13} & -\frac{2}{13} \end{pmatrix}$$

(5)

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 3 & -1 & 6 & 0 & 1 & 0 \\ -1 & 5 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{R_2 \to (-3)R_1 + R_2 \\ R_3 \to R_1 + R_3}]{} \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -10 & -6 & -3 & 1 & 0 \\ 0 & 8 & 5 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 \to R_2 + R_3]{} \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -2 & -1 & -2 & 1 & 1 \\ 0 & 8 & 5 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 \to (-\frac{1}{2}) R_2]{} \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 8 & 5 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{R_1 \to R_1 + (-3) R_2 \\ R_3 \to (-8) R_1 + R_3}]{} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{5}{2} & -2 & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 8 & -4 & -4 \end{array}\right)$$

$$\xrightarrow[\substack{R_1 \to R_1 + (-\frac{5}{2}) R_3 \\ R_2 \to R_2 + (-\frac{1}{2}) R_3}]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -22 & -\frac{17}{2} & -\frac{17}{2} \\ 0 & 1 & 0 & -3 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 1 & 8 & -4 & -4 \end{array}\right)$$