(1)

$$\begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{pmatrix} \xrightarrow[\substack{R_2 \to (-2)R_1 + R_2 \\ R_3 \to (-3)R_1 + R_3}]{} \begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 5 & -12 & 2 \end{pmatrix}$$

$$\xrightarrow[R_3 \to (-5)R_2 + 3R_3]{} \begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & -6 & 1 \end{pmatrix}$$

$$\begin{array}{rrrrrr} & 0 & 0 & 15 & -36 & 6 \\ -) & 0 & 0 & 15 & -30 & 5 \\ \hline & 0 & 0 & 0 & -6 & 1 \end{array}$$

$$\xrightarrow[\substack{R_2 \to \frac{1}{3} R_2 \\ R_3 \to -\frac{1}{6} R_3}]{} \begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & -\frac{1}{6} \end{pmatrix}$$

$$\xrightarrow[\substack{R_1 \to R_1 + (-2)R_3 \\ R_2 \to R_2 + 2R_3}]{} \begin{pmatrix} 1 & 2 & -1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{6} \end{pmatrix}$$

$$\xrightarrow[R_1 \to R_1 + R_2]{} \begin{pmatrix} 1 & 2 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{6} \end{pmatrix}$$

① Each row elementary op. corresponds to

$P_j \cdot X$ an elementary matrix

② P_j : regular

③ $P_1 \cdots P_l$: regular if $P_1, \ldots, P_l$ are regular.

An application – Linear Independence and Dependence

$$A=\begin{pmatrix}1&2&-3&0\\2&4&-2&2\\3&6&-4&3\end{pmatrix}\to\cdots\to B=\begin{array}{c}\begin{array}{cccc}x&y&z&w\end{array}\\\begin{pmatrix}1&2&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\end{array}$$

- $\exists P$ a regular squre matrix of size 3 satisfying $PA=B$.
- $PA=B\Leftrightarrow P\vec{a}_1=\vec{b}_1,\ P\vec{a}_2=\vec{b}_2,\ P\vec{a}_3=\vec{b}_3,\ P\vec{a}_4=\vec{b}_4$
- Since P is regular, it follows that

$$\vec{a}_1=P^{-1}\vec{b}_1,\ \vec{a}_2=P^{-1}\vec{b}_2,\ \vec{a}_3=P^{-1}\vec{b}_3,\ \vec{a}_4=P^{-1}\vec{b}_4$$

Accordingly we have the equivalence

$$c_1\vec{a}_1+c_2\vec{a}_2+c_3\vec{a}_3+c_4\vec{a}_4=\vec{0}\ \underset{P^{-1}}{\overset{P\cdot}{\Leftrightarrow}}\ c_1\vec{b}_1+c_2\vec{b}_2+c_3\vec{b}_3+c_4\vec{b}_4=\vec{0}$$

$\vec{a}_1, \vec{a}_2, \dots, \vec{a}_\ell$ are ~~is~~ L.I.

$\Leftrightarrow \left(c_1\vec{a}_1+c_2\vec{a}_2+\cdots+c_\ell\vec{a}_\ell=\vec{0} \Rightarrow c_1=c_2=\cdots=c_\ell=0\right)$

———— are L.D

$\Leftrightarrow \exists c_j\neq 0 \quad c_1\vec{a}_1+\cdots+c_\ell\vec{a}_\ell=\vec{0}$.

An application – Linear Independence and Dependence(2)

$\cancel{0=0}\quad 2\vec{a}_1-\vec{a}_2+0\vec{a}_3+0\vec{a}_4=\vec{0}$

- $\vec{a}_1,\vec{a}_2,\vec{a}_3,\vec{a}_4$ are linearly dependent. In fact

$$2\vec{a}_1-\vec{a}_2=\vec{0}\quad\Leftrightarrow 2\vec{b}_1-\vec{b}_2=\vec{0}$$

- $\vec{a}_1,\vec{a}_3,\vec{a}_4$ are linearly independent. In fact

$$c_1\vec{a}_1+c_3\vec{a}_3+c_4\vec{a}_4=\vec{0}\Leftrightarrow c_1\vec{b}_1+c_3\vec{b}_3+c_4\vec{b}_4=\vec{0}$$

$// \quad c_1\begin{pmatrix}1\\0\\0\end{pmatrix}+c_3\begin{pmatrix}0\\1\\0\end{pmatrix}+c_4\begin{pmatrix}0\\0\\1\end{pmatrix}$

$$\Leftrightarrow\begin{pmatrix}c_1\\c_3\\c_4\end{pmatrix}=\vec{0}\Leftrightarrow c_1=c_3=c_4=0$$

An application – Linear Systems of equations

$$\left\{\begin{array}{rrrrl} x & +2y & -3z & & =0 \\ 2x & +4y & -2z & +2w & =0 \\ 3x & +6y & -4z & +3w & =0 \end{array}\right. \Leftrightarrow \quad A\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \vec{0}$$

$P\cdot$ $\quad \Updownarrow \quad$ $P^{-1}\cdot$

$$\left\{\begin{array}{rrrrl} x & +2y & & & =0 \\ & & z & & =0 \\ & & & w & =0 \end{array}\right. \Leftrightarrow \quad B\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \vec{0}$$

$-2t$

Put $y = t$. Then the solution is expressed by

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2t \\ t \\ 0 \\ 0 \end{pmatrix} = t \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow -2$$

$= x\vec{a}_1 + y\vec{a}_2 + z\vec{a}_3 - \vec{a}_4$

An application – Linear Systems of equations (2)

$$\left\{\begin{array}{rrrl} x & +2y & -3z & =0 \\ 2x & +4y & -2z & =2 \\ 3x & +6y & -4z & =3 \end{array}\right. \Leftrightarrow \quad A\begin{pmatrix} x \\ y \\ z \\ -1 \end{pmatrix} = \vec{0}$$

$P^{-1}\cdot$ $\quad \Updownarrow \quad$ $P\cdot$

$$\left\{\begin{array}{rrrl} x & +2y & & =0 \\ & & z & =0 \\ 0x & +0y & +0z & =1 \end{array}\right. \Leftrightarrow \quad B\begin{pmatrix} x \\ y \\ z \\ -1 \end{pmatrix} = \vec{0}$$

It follows that there exists no solution to the above system of equations.

An application – Calculation of inverse matrices

Theorem

Let $A \in M_n(\mathbf{R})$ be a regular square matrix. Then A^{-1} is regular, and

$$\left(A^{-1}\right)^{-1} = A$$

Proof Since A is regular, we have

$$AA^{-1} = A^{-1}A = I_n$$

This means that A^{-1} is regular and $\left(A^{-1}\right)^{-1} = A$.

$A \in M_n(\mathbb{R})$

A is regular

$\iff \exists X \in M_n(\mathbb{R})$

$AX = XA = I_n$

An application – Calculation of inverse matrices(2)

We calculatex the inverse of $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$.

$$(A|I_3) = \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 \to (-2)R_1 + R_2 \\ R_3 \to (-4)R_1 + R_3 \end{array}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right) \quad R_2 \to (-1) \times R_2$$

An application – Calculation of inverse matrices(3)

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & \boxed{-1} & -6 & 1 & 1 \end{array}\right) \qquad R_3 \to (-1) \times R_2 + R_3$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right) \qquad R_3 \to (-1) \times R_3$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right) \qquad \begin{array}{l} R_1 \to R_1 + (-2) \times R_3 \\ R_2 \to R_2 + (-1) \times R_3 \end{array}$$

$\shortparallel$

A^{-1}

An application – Calculation of inverse matrices(4)

It follows from the above sequence of row elementary operations that we have a regular $P \in M_3(\mathbf{R})$ satisfying

$$P(A|I_3) = (I_3|B) \quad \text{i.e.} \quad (PA|P) = (I_3|B)$$

It follows that

$$PA = I_3 \quad \text{and} \quad P = B$$

$$\overbrace{P^{-1}P}^{I_3} A = P^{-1} I_3$$

We multiply P^{-1} from the left to $PA = I_3$ to get

$$A = P^{-1}$$

This implies that A is regular and

$$A^{-1} = \left(P^{-1}\right)^{-1} = P = B = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}$$

An application – Calculation of inverse matrices(5)

We calculatex the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$.

$$(A|I_3) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 \to (-2)R_1 + R_2 \\ R_3 \to (-3)R_1 + R_3 \end{array}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right) \quad R_2 \leftrightarrow R_3$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right) \quad \begin{array}{l} R_2 \to (-1) \times R_2 \\ R_3 \to (-1) \times R_3 \end{array}$$

An application – Calculation of inverse matrices(5)

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -5 & 0 & 2 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right) \quad R_1 \to R_1 + (-2) \times R_2$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right) \quad \begin{array}{l} R_1 \to R_1 + 3 \times R_3 \\ R_2 \to R_2 + (-3) \times R_3 \end{array}$$

The above sequence of row elementary operations shows that

$$A^{-1} = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

A^{-1}.

Statistics and Dot Products

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Bivariate Data

We are given a set of bivariate data:

x	y
x_1	y_1
$\vdots$	$\vdots$
x_n	y_n

In this situation, first consider the arithmetic mean of x and y:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

Next consider the variance of x and y

$$V(x) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2, \quad V(y) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

Bivariate Data (2)

We also consider the covariance of x and y

$$C_{xy} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

Now we associate two vectors to the data:

$$\vec{x} = \frac{1}{\sqrt{n}}\begin{pmatrix} x_1 - \bar{x} \\ \vdots \\ x_n - \bar{x} \end{pmatrix}, \quad \vec{y} = \frac{1}{\sqrt{n}}\begin{pmatrix} y_1 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{pmatrix}$$

to express the variance and the covariance by

$$V(x) = ||\vec{x}||^2, \quad V(y) = ||\vec{y}||^2, \quad C_{xy} = (\vec{x}, \vec{y})$$

Moreover the correlation coefficient of x and y is expressed by

$$\rho_{xy} := \frac{C_{xy}}{\sqrt{V(x)}\sqrt{V(y)}} = \frac{(\vec{x}, \vec{y})}{||\vec{x}|| \cdot ||\vec{y}||}$$

Bivariate Data (3)—Cauchy-Schwartz inequality

Recall the Cauchy-Schwartz inequality:

$$|(\vec{a}, \vec{b})| \leq ||\vec{a}|| \cdot ||\vec{b}|| \quad (\vec{a}, \vec{b} \in \mathbf{R}^n)$$

It follows that

$$-1 \leq \rho_{xy} \leq 1$$

We introduce a model to figure out what happens when $\rho_{xy} \to \pm 1$.

$$y = ax + b$$

We try to fit the model to the given data in the following way.

Bivariate Data (4)—Regression LIne

First introduce a variable

$$\varepsilon = y - (ax + b), \quad \text{namely} \quad \varepsilon_j = y_j - (ax_j + b) \quad (j = 1, \ldots, n)$$

The variable ε means the error of the data based on the model and we set up the coefficients a and b so that

- **(i)** $\bar{\varepsilon} = 0$,
- **(ii)** $V(\varepsilon)$ is minimized.

y_j: observed data for y

$ax_j + b$: theoretical value base on the model.

Bivariate Data (5)—Regression Line

The condition (i) is equivalent to

$$\bar{\varepsilon} = \bar{y} - a\bar{x} - b = 0$$

To look into the condition (ii), we introduce a vector $\vec{\varepsilon}$ by

$$\vec{\varepsilon} = \frac{1}{\sqrt{n}} \begin{pmatrix} \varepsilon_1 - \bar{\varepsilon} \\ \vdots \\ \varepsilon_n - \bar{\varepsilon} \end{pmatrix}$$

Moreover we have

$$\varepsilon_j - \bar{\varepsilon} = (y_j - ax_j - b) - \overbrace{(\bar{y} - \bar{x} - b)}^{\bar{\varepsilon}} = (y_j - \bar{y}) - a(x_j - \bar{x})$$

a

Then it follows that

$$\vec{\varepsilon} = \vec{y} - a\vec{x}$$

Bivariate Data (6)—Regression Line

Now we can minimize $V(\varepsilon)$ by $= \|\vec{\varepsilon}\|^2$

$$
\begin{aligned}
V(\varepsilon) &= \|\vec{y} - a\vec{x}\|^2 \\
&= \|\vec{y}\|^2 - 2a(\vec{x},\vec{y}) + a^2\|\vec{x}\|^2 \\
&= \|\vec{x}\|^2 \left(a - \frac{(\vec{x},\vec{y})}{\|\vec{x}\|^2}\right)^2 + \|\vec{y}\|^2 - \frac{(\vec{x},\vec{y})^2}{\|\vec{x}\|^2} \\
&\geq \|\vec{y}\|^2 - \frac{(\vec{x},\vec{y})^2}{\|\vec{x}\|^2}
\end{aligned}
$$

The equality holds at the end of the line when

$$a = \frac{(\vec{x},\vec{y})}{\|\vec{x}\|^2}$$

Bivariate Data (7)—Regression Line

Regression line

The line $y = ax + b$ is called the regression line when

$$a = \frac{(\vec{x},\vec{y})}{\|\vec{x}\|^2} = \frac{C_{xy}}{V(x)}, \quad b = \bar{y} - a\bar{x}$$

In the case of regression line we have

$$
\begin{aligned}
V(\varepsilon) &= \|\vec{y}\|^2 \left(1 - \frac{(\vec{x},\vec{y})^2}{\|\vec{y}\|^2 \cdot \|\vec{x}\|^2}\right) \\
&= V(y)\left(1 - \rho_{xy}^2\right)
\end{aligned}
$$

Moreover since $\bar{\varepsilon} = 0$, we have

$$V(\epsilon) = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i^2$$

$= \frac{1}{n}\sum_{i=1}^{n}(\varepsilon_i - \bar{\varepsilon})^2$

$\bar{\varepsilon} = 0$

Bivariate Data (7)—Regression Line

The sum $\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2$ is called the residual squre and it approaches to 0 when $\rho_{xy} \to \pm 1$