I

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Since A is regular we have the identity

$$A A^{-1} = A^{-1} A = I_n$$

Transposé the all side to get

$$^{t}(A^{-1})\, ^{t}A = {}^{t}A\, ^{t}(A^{-1}) = I_n$$

This implies that ^{t}A is regular and that

$$(^{t}A)^{-1} = {}^{t}(A^{-1})$$

II Since A is regular we have the identity

$$A A^{-1} = A^{-1} A = I_n.$$

Transpose the all side to get

$$^{t}(A^{-1})\, ^{t}A = {}^{t}A\, ^{t}(A^{-1}) = I_n.$$

we use $^{t}A = A$ to get

$$^{t}(A^{-1})\, A = I_n.$$

Multiply A^{-1} fr the right to get

$$^{t}(A^{-1}) = A^{-1}.$$

This meanse That A^{-1} is symmetric.

III

(1)

$$A = \begin{pmatrix} 0 & 1 & 3 & 2 \\ 1 & 2 & 2 & 0 \\ 2 & 4 & 4 & 3 \\ -1 & -1 & 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 2 & 4 & 4 & 3 \\ -1 & -1 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow[\substack{R_3 \to (-2)R_1 + R_3 \\ R_4 \to R_1 + R_3}]{} \begin{pmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 3 & 0 \end{pmatrix}$$

$$\xrightarrow[\substack{R_1 \to R_1 + (-2)R_2 \\ R_3 \to (-1)R_2 + R_3}]{} \begin{pmatrix} 1 & 0 & -4 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$$\xrightarrow[R_3 \to \frac{1}{3} R_3]{} \begin{pmatrix} 1 & 0 & -4 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$$\xrightarrow[\substack{R_1 \to R_1 + (-4)R_3 \\ R_2 \to R_2 + (-2)R_3 \\ R_4 \to 2R_3 + R_4}]{} \begin{pmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(ii)

$$R_2 \to (-3)R_1 + R_2, \quad R_4 \to (-3)R_1 + R_4$$

$$A = \begin{pmatrix} 1 & 2 & -1 & 4 \\ 3 & 2 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 3 & 3 & 3 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & -4 & 3 & -10 \\ 0 & 1 & 3 & 2 \\ 0 & -3 & 6 & -8 \end{pmatrix}$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{} \begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -4 & 3 & -10 \\ 0 & -3 & 6 & -8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 15 & -2 \\ 0 & 0 & 15 & -2 \end{pmatrix}$$

$$R_1 \to R_1 + (-2)R_2$$
$$R_3 \to 4R_2 + R_3$$
$$R_4 \to 3R_2 + R_4$$

$$\xrightarrow[R_3 \to \frac{1}{15}R_3]{} \begin{pmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & -\frac{2}{15} \\ 0 & 0 & 1 & -\frac{2}{15} \end{pmatrix}$$

$$\xrightarrow[\substack{R_1 \to R_1 + 7R_3 \\ R_2 \to R_2 + (-3)R_3 \\ R_4 \to -R_3 + R_4}]{} \begin{pmatrix} 1 & 0 & -7 & -\frac{14}{15} \\ 0 & 1 & 0 & \frac{12}{5} \\ 0 & 0 & 1 & -\frac{2}{15} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

IV (1)

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 3 \\ 3 & 3 & 1 & 4 \end{pmatrix} \xrightarrow[R_3 \to (-3)R_1 + R_3]{R_2 \to (-2)R_1 + R_2} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -3 & 1 & 1 \\ 0 & -3 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 \to -\frac{1}{3}R_2} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -3 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 \to 3R_2 + R_3]{R_1 \to R_1 + (-2)R_2} \begin{pmatrix} 1 & 0 & \frac{2}{3} & \frac{5}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It follows from the above sequence of row elementary operations that

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 3 \\ 3 & 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \vec{0} \iff \begin{cases} x \quad + \frac{2}{3} z + \frac{5}{3} w = 0 \\ y - \frac{1}{3} z - \frac{1}{3} w = 0. \end{cases}$$

Let $z = s$ and $w = t$, Then

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}s - \frac{5}{3}t \\ \frac{1}{3}s + \frac{1}{3}t \\ s \\ t \end{pmatrix}$$

(2)

$$\begin{pmatrix} 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 2 \\ 4 & -12 & 3 & 5 \end{pmatrix} \xrightarrow[R_3 \to (-4)R_1 + R_3]{R_2 \to (-2)R_1 + R_2} \begin{pmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 3 & -3 \end{pmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{pmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -3 \end{pmatrix} \xrightarrow{R_3 \to (-3)R_2 + R_3} \begin{pmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It follows from the above sequence of row elementary operations that

$$\begin{pmatrix} 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 2 \\ 4 & -12 & 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \vec{0} \iff \begin{cases} x - 3y \quad + 2w = 0 \\ \quad z - w = 0 \end{cases}$$

Let $y = s$, $w = t$. Then

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3s - 2t \\ s \\ t \\ t \end{pmatrix}.$$