

# Echelon Matrices and Row Elementary Operations

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# Echelon Matrices

## Definition

A matrix  $A = (a_{ij})$  is called in *Echelon* form if

- (i) there exist *nonzero* entries

$$a_{1j_1}, a_{2j_2}, \dots, a_{rj_r} \quad \text{where} \quad j_1 < j_2 < \dots < j_r$$

- (ii) with the property that

$$a_{ij} = 0 \quad \text{if} \quad j < j_i \text{ OR } i > r$$

$a_{1j_1}, a_{2j_2}, \dots, a_{rj_r}$  are called the distinguished elements of  $A$ .

## Example

$$\begin{pmatrix} \boxed{2} & 3 & 2 & 0 & 4 & 5 & -6 \\ 0 & 0 & \boxed{7} & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{6} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

# Echelon Matrices (2)

## Example

$$\begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \boxed{1} & 3 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

## Definition

An echelon matrix is called a row reduced echelon matrix if the distinguished elements are

- (i) the only nonzero entries in their respective columns
- (ii) each equal to 1

# Row Equivalence

## Definition

A matrix  $A$  is row equivalent to  $B$  if

$$A \rightarrow \cdots \rightarrow B$$

$B$  is obtained from  $A$  by a finite sequences of the following operations called *row elementary operations*.

- (i) Interchange of the  $i$ th row and  $j$ th row:  $R_i \leftrightarrow R_j$
- (ii) Multiply the  $i$ th row by a nonzero scalar  $\lambda \neq 0$ :  
 $R_i \rightarrow \lambda \rightarrow \lambda R_i$
- (iii) Replace the  $i$ th row by  $\lambda$  times the  $j$ th row plus the  $i$ th row:  $R_i \rightarrow R_i + \lambda R_j$ .

# Relation to Elementary Matrices

## Fact

A row elementary operation is identical to multiplication by an elementary matrix from the left.

Let us illustrate this fact by examples.

- (i)  $R_i \leftrightarrow R_j$

$$R_1 \leftrightarrow R_3 \Leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_3 \\ \mathbf{a}_2 \\ \mathbf{a}_1 \end{pmatrix}$$

- (ii)  $R_i \rightarrow \lambda R_i (\lambda \neq 0)$

$$R_2 \rightarrow \lambda R_2 \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 \\ \lambda \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix}$$

## Relation to Elementary Matrices(2)

- (iii)  $R_i \rightarrow \lambda R_j + R_i$

$$R_3 \rightarrow \lambda R_1 + R_3 \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \lambda \mathbf{a}_1 + \mathbf{a}_3 \end{pmatrix}$$

# Row Elementary Operations are invertible

- (i)  $R_i \leftrightarrow R_j$

$$(R_1 \leftrightarrow R_3)^{-1} = (R_1 \leftrightarrow R_3) \Leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- (ii)  $R_i \rightarrow \lambda R_i \ (\lambda \neq 0)$

$$(R_2 \rightarrow \lambda R_2)^{-1} = (R_2 \rightarrow \frac{1}{\lambda} R_2) \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Row Elementary Operations are invertible(2)

- (iii)  $R_i \rightarrow \lambda R_j + R_i$

$$\begin{aligned}(R_3 \rightarrow \lambda R_1 + R_3)^{-1} &= (R_3 \rightarrow -\lambda R_1 + R_3) \\ \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix}^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\lambda & 0 & 1 \end{pmatrix}\end{aligned}$$



# Row Equivalence–Example

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 5 & 3 \end{pmatrix} \quad \begin{array}{l} R_2 \rightarrow -2R_1 + R_2 \\ R_3 \rightarrow -3R_1 + R_3 \end{array}$$
$$\longrightarrow \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad R_3 \rightarrow -5R_2 + 4R_3$$

**Remark**  $(R_3 \rightarrow 4R_3) + (R_3 \rightarrow -5R_2 + R_3) = -5R_2 + 4R_3$

Two Elementary row operations are combined

- **(ii)+(iii)** Replace the  $i$ th row by the  $\lambda$  times  $j$ th row +  $\mu$  times  $i$ th row ( $\mu \neq 0$ ):

$$(R_i \rightarrow \mu \times R_i) + (R_i \rightarrow \lambda R_j + R_i) = R_i \rightarrow \lambda R_j + \mu R_i$$

## Row Equivalence–Example(2)

$$\begin{aligned} \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} &\longrightarrow \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} & R_3 \rightarrow \frac{1}{2}R_3 \\ &\longrightarrow \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & R_2 \rightarrow R_2 + (-2)R_3 \\ &\longrightarrow \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & R_2 \rightarrow \frac{1}{4}R_2 \\ &\longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & R_1 \rightarrow R_1 + 3R_2 \end{aligned}$$

# Theorem on Row Equivalence

## Theorem

Let  $A$  be a  $m \times n$  matrix. Then  $A$  is row equivalent to a unique row reduced echelon matrix. The number of its distinguished elements is called the rank of  $A$  denoted by  $\text{rank}(A)$ .

**Remark** The uniqueness in the theorem is difficult and complicated to show.

# Theorem on Row Equivalence

## Theorem

Let  $A$  be a matrix of type  $m \times n$ . If  $A$  is row equivalent to another matrix  $B$  of type  $m \times n$ . Then there exists a regular square matrix  $P$  of size  $m$  satisfying

$$PA = B$$

The proof of the theorem is based on the following two theorems.

- **Theorem** If  $P_1, P_2, \dots, P_\ell$  are regular square matrix of size  $m$ , then  $P = P_1 \dots P_\ell$  is regular.
- **Theorem** Elementary matrices are regular.

# An application – Linear Independence and Dependence

$$A = \begin{pmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 3 \end{pmatrix} \rightarrow \cdots \rightarrow B = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- $\exists P$  a regular square matrix of size 3 satisfying  $PA = B$ .
- $PA = B \Leftrightarrow P\vec{a}_1 = \vec{b}_1, P\vec{a}_2 = \vec{b}_2, P\vec{a}_3 = \vec{b}_3, P\vec{a}_4 = \vec{b}_4$
- Since  $P$  is regular, it follows that

$$\vec{a}_1 = P^{-1}\vec{b}_1, \vec{a}_2 = P^{-1}\vec{b}_2, \vec{a}_3 = P^{-1}\vec{b}_3, \vec{a}_4 = P^{-1}\vec{b}_4$$

Accordingly we have the equivalence

$$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 + c_4\vec{a}_4 = \vec{0} \Leftrightarrow c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 + c_4\vec{b}_4 = \vec{0}$$

# An application – Linear Independence and Dependence(2)

- $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$  are linearly dependent. In fact

$$2\vec{a}_1 - \vec{a}_2 = \vec{0} \Leftrightarrow 2\vec{b}_1 - \vec{b}_2 = \vec{0}$$

- $\vec{a}_1, \vec{a}_3, \vec{a}_4$  are linearly independent. In fact

$$c_1\vec{a}_1 + c_3\vec{a}_3 + c_4\vec{a}_4 = \vec{0} \Leftrightarrow c_1\vec{b}_1 + c_3\vec{b}_3 + c_4\vec{b}_4 = \vec{0}$$

$$\Leftrightarrow \begin{pmatrix} c_1 \\ c_3 \\ c_4 \end{pmatrix} = \vec{0} \Leftrightarrow c_1 = c_3 = c_4 = 0$$

# An application – Linear Systems of equations

$$\begin{cases} x + 2y - 3z = 0 \\ 2x + 4y - 2z + 2w = 0 \\ 3x + 6y - 4z + 3w = 0 \end{cases} \Leftrightarrow A \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \vec{0}$$
$$\Downarrow$$
$$\begin{cases} x + 2y = 0 \\ z = 0 \\ w = 0 \end{cases} \Leftrightarrow B \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \vec{0}$$

Put  $y = t$ . Then the solution is expressed by

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2t \\ t \\ 0 \\ 0 \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

# An application – Linear Systems of equations (2)

$$\begin{cases} x + 2y - 3z = 0 \\ 2x + 4y - 2z = 2 \\ 3x + 6y - 4z = 3 \end{cases} \Leftrightarrow A \begin{pmatrix} x \\ y \\ z \\ -1 \end{pmatrix} = \vec{0}$$
$$\Updownarrow$$
$$\begin{cases} x + 2y = 0 \\ \phantom{x} \phantom{+2y} z = 0 \\ 0x + 0y + 0z = 1 \end{cases} \Leftrightarrow B \begin{pmatrix} x \\ y \\ z \\ -1 \end{pmatrix} = \vec{0}$$

It follows that there exists no solution to the above system of equations.



# An application – Calculation of inverse matrices

## Theorem

Let  $A \in M_n(\mathbf{R})$  be a regular square matrix. Then  $A^{-1}$  is regular, and

$$(A^{-1})^{-1} = A$$

**Proof** Since  $A$  is regular, we have

$$AA^{-1} = A^{-1}A = I_n$$

This means that  $A^{-1}$  is regular and  $(A^{-1})^{-1} = A$ .

# An application – Calculation of inverse matrices(2)

We calculate the inverse of  $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$ .

$$(A|I_3) = \left( \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array} \right) \quad \begin{array}{l} R_2 \rightarrow (-2)R_1 + R_2 \\ R_3 \rightarrow (-4)R_1 + R_3 \end{array}$$

$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array} \right) \quad R_2 \rightarrow (-1) \times R_2$$

## An application – Calculation of inverse matrices(3)

$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array} \right) \quad R_3 \rightarrow (-1) \times R_2 + R_3$$

$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array} \right) \quad R_3 \rightarrow (-1) \times R_3$$

$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array} \right) \quad \begin{array}{l} R_1 \rightarrow R_1 + (-2) \times R_3 \\ R_2 \rightarrow R_2 + (-1) \times R_3 \end{array}$$

## An application – Calculation of inverse matrices(4)

It follows from the above sequence of row elementary operations that we have a regular  $P \in M_3(\mathbf{R})$  satisfying

$$P(A|I_3) = (I_3|B) \quad \text{i.e.} \quad (PA|P) = (I_3|B)$$

It follows that

$$PA = I_3 \quad \text{and} \quad P = B$$

We multiply  $P^{-1}$  from the left to  $PA = I_3$  to get

$$A = P^{-1}$$

This implies that  $A$  is regular and

$$A^{-1} = (P^{-1})^{-1} = P = B = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}$$

# An application – Calculation of inverse matrices(5)

We calculate the inverse of  $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ .

$$(A|I_3) = \left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array} \right) \quad \begin{array}{l} R_2 \rightarrow (-2)R_1 + R_2 \\ R_3 \rightarrow (-3)R_1 + R_3 \end{array}$$

$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array} \right) \quad R_2 \leftrightarrow R_3$$

$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array} \right) \quad \begin{array}{l} R_2 \rightarrow (-1) \times R_2 \\ R_3 \rightarrow (-1) \times R_3 \end{array}$$

## An application – Calculation of inverse matrices(5)

$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & -3 & -5 & 0 & 2 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array} \right) \quad R_1 \rightarrow R_1 + (-2) \times R_2$$

$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array} \right) \quad \begin{array}{l} R_1 \rightarrow R_1 + 3 \times R_3 \\ R_2 \rightarrow R_2 + (-3) \times R_3 \end{array}$$

The above sequence of row elementary operations shows that

$$A^{-1} = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$