Echelon Matrices and Row Elementary Operations

Nobuyuki TOSE

November 22, 2016

Echelon Matrices

Definition

A matrix $A = (a_{ij})$ is called in *Echelon* form if

- **(i)** there exist *nonzero* entiries

$$a_{1j_1}, a_{2j_2}, \ldots, a_{rj_r} \quad \text{where} \quad j_1 < j_2 < \cdots < j_r$$

- **(ii)** with the property that

$$a_{ij} = 0 \quad \text{if} \quad j < j_i \text{ OR } i > r$$

$a_{1j_1}, a_{2j_2}, \ldots, a_{rj_r}$ are called the distinguished elements of A.

Example

$$\begin{pmatrix} \boxed{2} & 3 & 2 & 0 & 4 & 5 & -6 \\ 0 & 0 & \boxed{7} & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{6} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Echelon Matrices (2)

Example

$$\begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \boxed{1} & 3 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

Definition

An echelon matrix is called a row reduced echelon matrix if the distinguished elements are

- **(i)** the only nonzero entries in their respective columns
- **(ii)** each equal to 1

Row Equivalence

Definition

A matrix A is row equivalent to B if

$$A \to \cdots \to B$$

B is obtained from A by a finite sequences of the following operations called *row elementary operations*.

- **(i)** Interchange of the ith row and jth row: $R_i \leftrightarrow R_j$
- **(ii)** Multiply the ith row by a nonzero scalar $\lambda \neq 0$: $R_i \to \lambda \to \lambda R_i$
- **(iii)** Replace the ith row by λ times the jth row plus the ith row: $R_i \to R_i + \lambda R_j$.

Relation to Elementary Matrices

Fact

A row elementary operation is identical to multiplication by an elementary matrix from the left.

Let us illustarte this fact by examples.

- **(i)** $R_i \leftrightarrow R_j$

$$R_1 \leftrightarrow R_3 \Leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_3 \\ \mathbf{a}_2 \\ \mathbf{a}_1 \end{pmatrix}$$

- **(ii)** $R_i \to \lambda R_i (\lambda \neq 0)$

$$R_2 \to \lambda R_2 \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 \\ \lambda \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix}$$

Relation to Elementary Matrices(2)

- **(iii)** $R_i \to \lambda R_j + R_i$

$$R_3 \to \lambda R_1 + R_3 \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \lambda \mathbf{a}_1 + \mathbf{a}_3 \end{pmatrix}$$

Row Elementary Operations are invertible

- **(i)** $R_i \leftrightarrow R_j$

$$(R_1 \leftrightarrow R_3)^{-1} = (R_1 \leftrightarrow R_3) \Leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- **(ii)** $R_i \to \lambda R_j \ (\lambda \neq 0)$

$$(R_2 \to \lambda R_2)^{-1} = (R_2 \to \frac{1}{\lambda} R_2) \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Row Elementary Operations are invertible(2)

- **(iii)** $R_i \to \lambda R_j + R_i$

$$(R_3 \to \lambda R_1 + R_3)^{-1} = (R_3 \to -\lambda R_1 + R_3)$$

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\lambda & 0 & 1 \end{pmatrix}$$

Row Equivalence–Example

$$\begin{pmatrix}1 & 2 & -3 & 0\\ 2 & 4 & -2 & 2\\ 3 & 6 & -4 & 3\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 2 & -3 & 0\\ 0 & 0 & 4 & 2\\ 0 & 0 & 5 & 3\end{pmatrix} \qquad \begin{matrix}R_2 \to -2R_1 + R_2\\ R_3 \to -3R_1 + R_3\end{matrix}$$

$$\longrightarrow \begin{pmatrix}1 & 2 & -3 & 0\\ 0 & 0 & 4 & 2\\ 0 & 0 & 0 & 2\end{pmatrix} \qquad R_3 \to -5R_2 + 4R_3$$

Remark $(R_3 \to 4R_3) + (R_3 \to -5R_2 + R_3) = -5R_2 + 4R_3$

Two Elementary row operations are combined

- **(ii)+(iii)** Replace the ith row by the λ times jth row + μ times ith row ($\mu \neq 0$):

$$(R_i \to \mu \times R_i) + (R_i \to \lambda R_j + R_i) = R_i \to \lambda R_j + \mu R_i$$

Row Equivalence–Example(2)

$$
\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_3 \to \frac{1}{2} R_3
$$

$$
\longrightarrow \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_2 \to R_2 + (-2) R_3
$$

$$
\longrightarrow \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_2 \to \frac{1}{4} R_2
$$

$$
\longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_1 \to R_1 + 3 R_2
$$

Theorem on Row Equivalence

Theorem

Let A be a $m \times n$ matrix. Then A is row equivalent to a unique row reduced echelon matrix. The number of its distinguished elements is called the rank of A denoted by $\mathrm{rank}(A)$.

Remark The uniqueness in the theorem is difficult and complicated to show.

Theorem on Row Equivalence

Theorem

Let A be a matrix of type $m \times n$. If A is row equivalennt to another matrix B of type $m \times n$. Then there exists a regular square matrix P of size m satisfying

$$PA = B$$

The proof of the theorem is based on the following two theorems.

- **Theorem** If $P_1,\ P_2, \ldots,\ P_\ell$ are regular squre matrix of size m, then $P = P_1 \ldots P_\ell$ is regular.
- **Theorem** Elementary matrices are regular.

An application – Linear Independence and Dependence

$$A = \begin{pmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 3 \end{pmatrix} \to \cdots \to B = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- $\exists P$ a regular squre matrix of size 3 satisfying $PA = B$.
- $PA = B \Leftrightarrow P\vec{a}_1 = \vec{b}_1,\ P\vec{a}_2 = \vec{b}_2,\ P\vec{a}_3 = \vec{b}_3,\ P\vec{a}_4 = \vec{b}_4$
- Since P is regular, it follows that

$$\vec{a}_1 = P^{-1}\vec{b}_1,\ \vec{a}_2 = P^{-1}\vec{b}_2,\ \vec{a}_3 = P^{-1}\vec{b}_3,\ \vec{a}_4 = P^{-1}\vec{b}_4$$

Accordingly we have the equivalence

$$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 + c_4\vec{a}_4 = \vec{0} \Leftrightarrow c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 + c_4\vec{b}_4 = \vec{0}$$

An application – Linear Independence and Dependence(2)

- $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ are linearly dependent. In fact

$$2\vec{a}_1 - \vec{a}_2 = \vec{0} \quad \Leftrightarrow 2\vec{b}_1 - \vec{b}_2 = \vec{0}$$

- $\vec{a}_1, \vec{a}_3, \vec{a}_4$ are linearly independent. In fact

$$\begin{aligned} c_1\vec{a}_1 + c_3\vec{a}_3 + c_4\vec{a}_4 = \vec{0} &\Leftrightarrow c_1\vec{b}_1 + c_3\vec{b}_3 + c_4\vec{b}_4 = \vec{0} \\ &\Leftrightarrow \begin{pmatrix} c_1 \\ c_3 \\ c_4 \end{pmatrix} = \vec{0} \Leftrightarrow c_1 = c_3 = c_4 = 0 \end{aligned}$$

An application – Linear Systems of equations

$$\left\{\begin{array}{rrrrl} x & +2y & -3z & & =0 \\ 2x & +4y & -2z & +2w & =0 \\ 3x & +6y & -4z & +3w & =0 \end{array}\right. \Leftrightarrow \quad A\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \vec{0}$$

$$\Updownarrow$$

$$\left\{\begin{array}{rrrrl} x & +2y & & & =0 \\ & & z & & =0 \\ & & & w & =0 \end{array}\right. \Leftrightarrow \quad B\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \vec{0}$$

Put $y = t$. Then the solution is expressed by

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2t \\ t \\ 0 \\ 0 \end{pmatrix} = t\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

An application – Linear Systems of equations (2)

$$\left\{\begin{array}{rrrl} x & +2y & -3z & = 0 \\ 2x & +4y & -2z & = 2 \\ 3x & +6y & -4z & = 3 \end{array}\right. \Leftrightarrow \qquad A\begin{pmatrix} x \\ y \\ z \\ -1 \end{pmatrix} = \vec{0}$$

$$\Updownarrow$$

$$\left\{\begin{array}{rrrl} x & +2y & & = 0 \\ & & z & = 0 \\ 0x & +0y & +0z & = 1 \end{array}\right. \Leftrightarrow \qquad B\begin{pmatrix} x \\ y \\ z \\ -1 \end{pmatrix} = \vec{0}$$

It follows that there exists no solution to the above system of equations.