

Matrices and their operations No. 3

Transposition of $m \times n$ Matrices

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Transposition of vectors

Definition

$${}^t \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = (a_1 \dots a_n)$$

$${}^t(a_1 \dots a_n) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Transposition of vectors – Basic Properties

Basic Properties (1)

- (i) $\vec{a}, \vec{b} \in \mathbf{R}^n$, we have

$${}^t(\vec{a} + \vec{b}) = {}^t\vec{a} + {}^t\vec{b}, \quad {}^t(\lambda\vec{a}) = \lambda {}^t\vec{a}$$

- (ii) $\mathbf{a}, \mathbf{b} \in (\mathbf{R}^n)^*$, we have

$${}^t(\mathbf{a} + \mathbf{b}) = {}^t\mathbf{a} + {}^t\mathbf{b}, \quad {}^t(\lambda\mathbf{a}) = \lambda {}^t\mathbf{a}$$

Transposition of vectors – Basic Properties

Basic Properties (2)

- (iii) $\vec{a}_1, \dots, \vec{a}_n \in \mathbf{R}^m$, we have

$${}^t(c_1\vec{a}_1 + \dots + c_n\vec{a}_n) = c_1{}^t\vec{a}_1 + \dots + c_n{}^t\vec{a}_n$$

It follows from the basic property (iii) that

$${}^t((\vec{a}_1 \ \dots \ \vec{a}_n)\vec{c}) = (c_1 \ \dots \ c_n) \begin{pmatrix} {}^t\vec{a}_1 \\ \vdots \\ {}^t\vec{a}_n \end{pmatrix}$$

Transposition of vectors – Basic Properties

The transposition of vectors is related to the dot product by the following basic property (iv).

Basic Properties (3)

- **(iv)** For $\vec{a}, \vec{b} \in \mathbf{R}^n$, we have

$$(\vec{a}, \vec{b}) = {}^t\vec{a} \cdot \vec{b}$$

Remark that the both sides are equal to

$$a_1 b_1 + \cdots + a_n b_n$$

Transposition of $m \times n$ Matrices

Definition

$$A = (\vec{a}_1 \ \dots \ \vec{a}_n) = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

Then the transposition of A is a $n \times m$ matrix defined by

$${}^tA = ({}^t\mathbf{a}_1 \ \dots \ {}^t\mathbf{a}_n) = \begin{pmatrix} {}^t\vec{a}_1 \\ \vdots \\ {}^t\vec{a}_m \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{i1} & \dots & a_{m1} \\ \vdots & & \vdots & & \vdots \\ a_{1j} & \dots & a_{ij} & \dots & a_{mj} \\ \vdots & & \vdots & & \vdots \\ a_{1n} & \dots & a_{in} & \dots & a_{mn} \end{pmatrix}$$



Why do we need the transposition of matrices?

Why do we need such a complicated operation for matrices? We can explain it by the following theorem.

Theorem

Given a $m \times n$ matrix A . Then we have

$$(A\vec{x}, \vec{y}) = (\vec{x}, {}^tA\vec{y})$$

for $\vec{x} \in \mathbf{R}^n$ and $\vec{y} \in \mathbf{R}^m$.

Before giving proof for the theorem, remark that $A\vec{x} \in \mathbf{R}^m$ and that ${}^tA\vec{y} \in \mathbf{R}^n$.

Why we do need the transposition of matrices? (2)

Proof.

$$\begin{aligned} LHS &= (x_1 \vec{a}_1 + \cdots + x_n \vec{a}_n, \vec{y}) \\ &= x_1(\vec{a}_1, \vec{y}) + \cdots + x_n(\vec{a}_n, \vec{y}) \\ &= \left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} (\vec{a}_1, \vec{y}) \\ \vdots \\ (\vec{a}_n, \vec{y}) \end{pmatrix} \right) = \left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} {}^t \vec{a}_1 \vec{y} \\ \vdots \\ {}^t \vec{a}_n \vec{y} \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} {}^t \vec{a}_1 \\ \vdots \\ {}^t \vec{a}_n \end{pmatrix} \vec{y} \right) = (\vec{x}, {}^t A \vec{y}) = RHS \end{aligned}$$



2 dimensional linear subspaces and a problem

Given $\vec{a}, \vec{b} \in \mathbf{R}^n$ and assume that

$$\vec{a} \nparallel \vec{b}$$

We define the 2-dimensional linear subspace V generated by $\vec{a}$ and $\vec{b}$ by

$$V := \{s_1\vec{a} + s_2\vec{b}; s_1, s_2 \in \mathbf{R}\}$$

Problem

Given $\vec{c} \in \mathbf{R}^n$. Then consider the problem to find $\vec{v}_0 \in V$ satisfying

$$(\#) \quad \|\vec{c} - \vec{v}_0\|^2 \leq \|\vec{c} - \vec{v}\|^2 \quad (\vec{v} \in V)$$

Namely to find $\vec{v}_0 \in V$ minimizing

$$\|\vec{c} - \vec{v}\|^2 \quad (\vec{v} \in V)$$

Solution to the problem (1)

The first step of the solution is given in the following theorem.

Theorem

If $\vec{v}_0 \in V$ satisfies the condition,

$$(\#\#) \quad (\vec{c} - \vec{v}_0, \vec{v}) = 0 \quad (\vec{v} \in V)$$

then the inequality (#) holds.

Take any $\vec{v} \in V$. Then

$$\begin{aligned} \|\vec{c} - \vec{v}\|^2 &= \|\vec{c} - \vec{v}_0 + (\vec{v}_0 - \vec{v})\|^2 \\ &= \|\vec{c} - \vec{v}_0\|^2 + \|\vec{v}_0 - \vec{v}\|^2 \geq \|\vec{c} - \vec{v}_0\|^2 \end{aligned}$$

Remark that the equality holds at the last inequality if

$$\|\vec{v}_0 - \vec{v}\|^2 = 0 \quad \text{i.e.} \quad \vec{v} = \vec{v}_0$$

Solution to the problem (2)

Now we try to find the vector $\vec{v}_0 \in V$ satisfying (##) by using a $n \times 2$ matrix $A = (\vec{a} \ \vec{b})$ and its tranposition tA . $\vec{v}_0, \vec{v}$ are expressed with A by

$$\vec{v} = A \begin{pmatrix} s \\ t \end{pmatrix}, \quad \vec{v}_0 = A \begin{pmatrix} s_0 \\ t_0 \end{pmatrix}$$

Then the condition (##) is equivalent to

$$(\vec{c} - A \begin{pmatrix} s_0 \\ t_0 \end{pmatrix}, A \begin{pmatrix} s \\ t \end{pmatrix}) = 0 \quad \left(\begin{pmatrix} s \\ t \end{pmatrix} \in \mathbf{R}^2 \right)$$

Moreover the condition is equivalent to

$$\left({}^tAA \begin{pmatrix} s_0 \\ t_0 \end{pmatrix} - {}^tA\vec{c}, \begin{pmatrix} s \\ t \end{pmatrix} \right) = 0 \quad \left(\begin{pmatrix} s \\ t \end{pmatrix} \in \mathbf{R}^2 \right)$$

Solution to the problem (3)

Since $\begin{pmatrix} s \\ t \end{pmatrix} \in \mathbf{R}^2$ is arbitrary, the condition is equivalent to

$${}^tAA \begin{pmatrix} s_0 \\ t_0 \end{pmatrix} = {}^tA\vec{c}$$

The matrix tAA is of type 2×2 . Moreover we will find that tAA is regular under the condition that $\vec{a} \nparallel \vec{b}$. Accordingly we have

$$\vec{v}_0 = ({}^tAA)^{-1} {}^tA\vec{c}$$

The Gram Matrix

tAA is called *Gram matrix* of A . In this case we have

$${}^tAA = \begin{pmatrix} {}^t\vec{a} \\ {}^t\vec{b} \end{pmatrix} (\vec{a} \ \vec{b}) = \begin{pmatrix} {}^t\vec{a}\vec{a} & {}^t\vec{a}\vec{b} \\ {}^t\vec{b}\vec{a} & {}^t\vec{b}\vec{b} \end{pmatrix} = \begin{pmatrix} \|\vec{a}\|^2 & (\vec{a}, \vec{b}) \\ (\vec{b}, \vec{a}) & \|\vec{b}\|^2 \end{pmatrix}$$

Transposition of vectors – Basic Properties

Basic Properties (2)

- (iii) $\vec{a}_1, \dots, \vec{a}_n \in \mathbf{R}^m$, we have

$${}^t(c_1\vec{a}_1 + \dots + c_n\vec{a}_n) = c_1{}^t\vec{a}_1 + \dots + c_n{}^t\vec{a}_n$$

It follows from the basic property (iii) that

$${}^t((\vec{a}_1 \ \dots \ \vec{a}_n)\vec{c}) = (c_1 \ \dots \ c_n) \begin{pmatrix} {}^t\vec{a}_1 \\ \vdots \\ {}^t\vec{a}_n \end{pmatrix}$$

Transposition of Matrices – Basic Properties(1)

Addition, Scalar Multiplication and Transposition

Let A, B be matrices of type $m \times n$. Then we have

$${}^t(A + B) = {}^tA + {}^tB, \quad {}^t(\lambda A) = \lambda({}^tA)$$

$$\begin{aligned} {}^t(A + B) &= {}^t(\vec{a}_1 + \vec{b}_1 \ \dots \ \vec{a}_n + \vec{b}_n) \\ &= \begin{pmatrix} {}^t(\vec{a}_1 + \vec{b}_1) \\ \vdots \\ {}^t(\vec{a}_n + \vec{b}_n) \end{pmatrix} = \begin{pmatrix} {}^t\vec{a}_1 + {}^t\vec{b}_1 \\ \vdots \\ {}^t\vec{a}_n + {}^t\vec{b}_n \end{pmatrix} \\ &= \begin{pmatrix} {}^t\vec{a}_1 \\ \vdots \\ {}^t\vec{a}_n \end{pmatrix} + \begin{pmatrix} {}^t\vec{b}_1 \\ \vdots \\ {}^t\vec{b}_n \end{pmatrix} = {}^tA + {}^tB \end{aligned}$$

Transposition of Matrices – Basic Properties(1)

$$\begin{aligned} {}^t(\lambda A) &= {}^t(\lambda \vec{a}_1 \ \dots \ \lambda \vec{a}_n) = \begin{pmatrix} {}^t(\lambda \vec{a}_1) \\ \vdots \\ {}^t(\lambda \vec{a}_n) \end{pmatrix} = \begin{pmatrix} \lambda {}^t\vec{a}_1 \\ \vdots \\ \lambda {}^t\vec{a}_n \end{pmatrix} \\ &= \lambda \begin{pmatrix} {}^t\vec{a}_1 \\ \vdots \\ {}^t\vec{a}_n \end{pmatrix} = \lambda {}^tA \end{aligned}$$

Transposition of Matrices – Basic Properties(2)

Multiplication of Matrices and their Transposition

Let A be a matrix of type $m \times n$, and X of type $n \times \ell$. Then we have

$${}^t(AX) = {}^tX {}^tA$$

$$\begin{aligned} {}^t(AX) &= {}^t(A\vec{x}_1 \ \dots \ A\vec{x}_\ell) = \begin{pmatrix} {}^t(A\vec{x}_1) \\ \vdots \\ {}^t(A\vec{x}_\ell) \end{pmatrix} \\ &= \begin{pmatrix} {}^t\vec{x}_1 {}^tA \\ \vdots \\ {}^t\vec{x}_\ell {}^tA \end{pmatrix} = \begin{pmatrix} {}^t\vec{x}_1 \\ \vdots \\ {}^t\vec{x}_\ell \end{pmatrix} {}^tA = {}^tX {}^tA \end{aligned}$$