$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \rightsquigarrow {}^t\vec{x} = (x_1 \cdots x_n)$$

$$\mathbb{x} = (x_1 \cdots x_n) \rightsquigarrow {}^t\mathbb{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$${}^t(\lambda \vec{x} + \mu \vec{y}) = \lambda\, {}^t\vec{x} + \mu\, {}^t\vec{y}$$

$${}^t(\lambda \mathbb{x} + \mu\, \mathbb{y}) = \lambda\, {}^t\mathbb{x} + \mu\, {}^t\mathbb{y}$$

$${}^t(\vec{x_1} + \vec{x_2} + \cdots + \vec{x_n}) = {}^t\vec{x_1} + \cdots + {}^t\vec{x_n}$$

$${}^t(c_1\vec{x_1} + \cdots + c_n\vec{x_n}) = {}^t(c_1\vec{x_1}) + \cdots + {}^t(c_n\vec{x_n})$$

$$= c_1\, {}^t\vec{x_1} + \cdots + c_n\, {}^t\vec{x_n}.$$

Transposition of vectors – Basic Properties

Basic Properties (2)

- **(iii)** $\vec{a}_1, \ldots, \vec{a}_n \in \mathbf{R}^m$, we have

$$^t(c_1\vec{a}_1 + \cdots + c_n\vec{a}_n) = c_1{}^t\vec{a}_1 + \cdots + c_n{}^t\vec{a}_n$$

It follows from the basic property (iii) that

$$^t((\vec{a}_1 \ \ldots \ \vec{a}_n)\vec{c}) = (c_1 \ \ldots \ c_n)\begin{pmatrix} {}^t\vec{a}_1 \\ \vdots \\ {}^t\vec{a}_n \end{pmatrix}$$

$$= {}^t\left((\vec{a}_1 \cdots \vec{a}_n)\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}\right)$$

Transposition of Matrices – Basic Properties(1)

Addition, Scalar Multiplication and Transposition

Let A, B be matrices of type $m \times n$. Then we have

$$^t(A+B) = {}^tA + {}^tB, \quad {}^t(\lambda A) = \lambda({}^tA)$$

$$\begin{aligned} {}^t(A+B) &= {}^t(\vec{a}_1 + \vec{b}_1 \ \ldots \ \vec{a}_n + \vec{b}_n) \\ &= \begin{pmatrix} {}^t(\vec{a}_1 + \vec{b}_1) \\ \vdots \\ {}^t(\vec{a}_n + \vec{b}_n) \end{pmatrix} = \begin{pmatrix} {}^t\vec{a}_1 + {}^t\vec{b}_1 \\ \vdots \\ {}^t\vec{a}_n + {}^t\vec{b}_n \end{pmatrix} \\ &= \begin{pmatrix} {}^t\vec{a}_1 \\ \vdots \\ {}^t\vec{a}_n \end{pmatrix} + \begin{pmatrix} {}^t\vec{b}_1 \\ \vdots \\ {}^t\vec{b}_n \end{pmatrix} = {}^tA + {}^tB \end{aligned}$$

$$A = \begin{pmatrix} a|_1 \\ \vdots \\ a|_m \end{pmatrix} = \begin{pmatrix} \vec{a_1} & \cdots & \vec{a_n} \end{pmatrix}$$

$m \times n$

with m rows

and with n columns.

$$^tA = \begin{pmatrix} {}^t a|_1 & {}^t a|_2 & \cdots & {}^t a|_n \end{pmatrix}$$

$$= \begin{pmatrix} {}^t\vec{a_1} \\ \vdots \\ {}^t\vec{a_n} \end{pmatrix}$$

with n rows

and with m columns.

of type $n \times m$.

Transposition of Matrices – Basic Properties(1)

$$
{}^t(\lambda A) = {}^t(\lambda\vec{a}_1 \ \dots \ \lambda\vec{a}_n) = \begin{pmatrix} {}^t(\lambda\vec{a}_1) \\ \vdots \\ {}^t(\lambda\vec{a}_n) \end{pmatrix} = \begin{pmatrix} \lambda\,{}^t\vec{a}_1 \\ \vdots \\ \lambda\,{}^t\vec{a}_n \end{pmatrix}
$$

$$
= \lambda \begin{pmatrix} {}^t\vec{a}_1 \\ \vdots \\ {}^t\vec{a}_n \end{pmatrix} = \lambda\,{}^tA
$$

Transposition of Matrices – Basic Properties(2)

Multiplication of Matrices and their Transposition

Let A be a matrix of type $m \times n$, and X of type $n \times \ell$. Then we have

$$
{}^t(AX) = {}^tX\,{}^tA
$$

$X = (\vec{x}_1 \ \cdots \ \vec{x}_\ell)$

$\vec{x}_j \in \mathbb{R}^n$

$\downarrow$

$A\vec{x}_j \in \mathbb{R}^m$

$$
{}^t(AX) = {}^t(A\vec{x}_1 \ \dots \ A\vec{x}_\ell) = \begin{pmatrix} {}^t(A\vec{x}_1) \\ \vdots \\ {}^t(A\vec{x}_\ell) \end{pmatrix}
$$

$$
= \begin{pmatrix} {}^t\vec{x}_1\,{}^tA \\ \vdots \\ {}^t\vec{x}_\ell\,{}^tA \end{pmatrix} = \begin{pmatrix} {}^t\vec{x}_1 \\ \vdots \\ {}^t\vec{x}_\ell \end{pmatrix} {}^tA = {}^tX\,{}^tA
$$

${}^t(A\vec{x}) = {}^t\vec{x}\ {}^tA.$

Echelon Matrices and Row Elementary Operations

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Echelon Matrices

Definition

A matrix $A = (a_{ij})$ is called in *Echelon* form if

- **(i)** there exist *nonzero* entiries

$$a_{1j_1}, a_{2j_2}, \ldots, a_{rj_r} \quad \text{where} \quad j_1 < j_2 < \cdots < j_r$$

- **(ii)** with the property that

$$a_{ij} = 0 \quad \text{if} \quad j < j_i \text{ OR } i > r$$

$a_{1j_1}, a_{2j_2}, \ldots, a_{rj_r}$ are called the distinguished elements of A

Example

$$\begin{pmatrix} \boxed{2} & 3 & 2 & 0 & 4 & 5 & -6 \\ 0 & 0 & \boxed{7} & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{6} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Echelon Matrices (2)

Example

$$\begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \boxed{1} & 3 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

$j_1 = 2 < j_2 = 4 < j_3 = 5 < j_4 = 7$.

A

n echelon matrix is called a row reduced echelon matrix if the distinguished elements are in the strict sense.

- **(i)** the only nonzero entries in their respective ~~rows~~ columns
- **(ii)** each equal to 1

Relation to Elementary Matrices

Fact

A row elementary operation is identical to multiplication by an elementary matrix from the left.

Let us illustarte this fact by examples.

- **(i)** $R_i \leftrightarrow R_j$

$$R_1 \leftrightarrow R_3 \Leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_3 \\ \mathbf{a}_2 \\ \mathbf{a}_1 \end{pmatrix}$$

- **(ii)** $R_i \to \lambda R_j (\lambda \neq 0)$

i

$$R_2 \to \lambda R_2 \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 \\ \lambda \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix}$$

Relation to Elementary Matrices

- **(iii)** $R_i \leftrightarrow \lambda R_j + R_i$

$$R_3 \leftrightarrow \lambda R_1 + R_3 \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \lambda \mathbf{a}_1 + \mathbf{a}_3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Row Elementary Operations are invertible

- **(i)** $R_i \leftrightarrow R_j$

$$(R_1 \leftrightarrow R_3)^{-1} = (R_1 \leftrightarrow R_3) \Leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- **(ii)** $R_i \to \lambda R_j \ (\lambda \neq 0)$

$$(R_2 \to \lambda R_2)^{-1} = (R_2 \to \frac{1}{\lambda} R_2) \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \lambda\mu & \\ & & 1 \end{pmatrix}$$

If $\lambda\mu = 1$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3.$$

Row Elementary Operations are invertible

Elementary Matrices are regular.

- **(iii)** $R_i \leftrightarrow \lambda R_j + R_i$

$$(R_3 \leftrightarrow \lambda R_1 + R_3)^{-1} = (R_3 \leftrightarrow -\lambda R_1 + R_3)$$

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\lambda & 0 & 1 \end{pmatrix}$$

Row Equivalence

Definition

A matrix A is row equivalent to B if

$$A \to \cdots \to B$$

B is obtained from A by a finite sequences of the following operations called *row elementary operations*.

- **(i)** Interchange of the ith row and jth row: $R_i \leftrightarrow R_j$
- **(ii)** Multiply the ith row by a nonzero scalar $\lambda \neq 0$: $R_i \to \lambda \to \lambda R_i$
- **(iii)** Replace the ith row by λ times the jth row plus the ith row: $R_i \to R_i + \lambda R_j$.

Row Equivalence–Example

0 0 20 12

$$
\begin{pmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 5 & 3 \end{pmatrix} \quad \begin{matrix} R_2 \to -2R_1 + R_2 \\ R_3 \to -3R_1 + R_3 \end{matrix}
$$

$$
\longrightarrow \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad R_3 \to -5R_2 + 4R_3
$$

Remark $(R_3 \to 4R_3) + (R_3 \to -5R_2 + R_3) = -5R_2 + 4R_3$

Two Elementary row operations are combined

- **(ii)+(iii)** Replace the ith row by the λ times jth row + μ times ith row ($\mu \neq 0$):

$$
(R_i \to \mu \times R_i) + (R_i \to \lambda R_j + R_i) = R_i \to \lambda R_j + \mu R_i
$$

Row Equivalence–Example(2)

$$
\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_3 \to \frac{1}{2} R_3
$$

$$
\longrightarrow \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_2 \to R_2 + (-2)R_3
$$

$$
\longrightarrow \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_2 \to \frac{1}{4} R_2
$$

$$
\longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_1 \to R_1 + 3R_2
$$

row reduced echelon matrix.

Theorem on Row Equivalence

Theorem

Let A be a $m \times n$ matrix. Then A is row equivalent to a unique row reduced echelon matrix. The number of its distinguished elements is called the rank of A denoted by $\mathrm{rank}(A)$.

Remark The uniqueness in the theorem is difficult and complicated to show.

$$\mathrm{rank}\begin{pmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 3 \end{pmatrix} = 3.$$

Theorem on Row Equivalence

Theorem

Let A be a matrix of type $m \times n$. If A is row equivalennt to another matrix B of type $m \times n$. Then there exists a regular square matrix P of size m satisfying

$$PA = B$$

The proof of the theorem is based on the following two theorems.

- **Theorem** If $P_1,\ P_2, \ldots,\ P_\ell$ are regular squre matrix of size m, then $P = P_1 \ldots P_\ell$ is regular.
- **Theorem** Elementary matrices are regular.

Th

P_1, P_2 are square matrices of size m.

We assume P_1, P_2 are regular.

Then $P_1 P_2$ is regular

$$(P_1 P_2)(P_2^{-1} P_1^{-1}) = P_1 (P_2 P_2^{-1}) P_1^{-1}$$
$$= (P_1 I_m) P_1^{-1} = P_1 P_1^{-1} = I_m$$

$$(P_2^{-1} P_1^{-1})(P_1 P_2) = P_2^{-1} (P_1^{-1} P_1) P_2$$
$$= P_2^{-1} (I_m P_2) = P_2^{-1} P_2 = I_m$$

$$\boxed{(P_1 P_2)^{-1} = P_2^{-1} P_1^{-1}}$$

$P_1 P_2$ is regular
P_3 ——
$\Big\} \to P_1 P_2 P_3$ is regular

$P_1 P_2 P_3$ ——
P_4 i——
$\Big\} \to P_1 P_2 P_3 P_4$ is regular.

$A \to \cdots \to B$

$P_\ell \cdots P_2 P_1 A = B$ $\quad P_j$ elementary

$P = P_\ell \cdots P_2 P_1$ Regular.

$PA = B$.