2016/11/08

I

(i)

$$A+B = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 0 & -3 \\ -1 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 1 & 7 \end{pmatrix}$$

$$3A-4B = 3\begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{pmatrix} - 4\begin{pmatrix} 4 & 0 & 3 \\ -1 & -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -3 & 6 \\ 0 & 9 & 16 \end{pmatrix} - \begin{pmatrix} 16 & 0 & 18 \\ -4 & -8 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} -13 & -3 & -12 \\ 4 & 17 & 4 \end{pmatrix}$$

(ii)

$$AC = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & -3 & 0 & 1 \\ 5 & -1 & -4 & 2 \\ -1 & 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -10 & -2 & 4 & 5 \\ 11 & -3 & 12 & 18 \end{pmatrix}$$

$$AD = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \end{pmatrix}$$

$$BC = \begin{pmatrix} 4 & 0 & -3 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 2 & -3 & 0 & 1 \\ 5 & -1 & -4 & 2 \\ -1 & 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -12 & 0 & -5 \\ -15 & 5 & 8 & 4 \end{pmatrix}$$

(iii) $${}^tA = \begin{pmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$B\,{}^tA = \begin{pmatrix} 4 & 0 & -3 \\ -1 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -2 & -12 \\ 5 & 6 \end{pmatrix}$$

$$D\,{}^tD = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}(2\ -1\ 3) = \begin{pmatrix} 4 & -2 & 6 \\ -2 & 1 & -3 \\ 6 & -3 & 9 \end{pmatrix}$$

$${}^tDD = (2\ -1\ 3)\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = (14)$$

II

$$e_1 A = (a_1 \; a_2 \; a_3 \; a_4)$$

$$e_2 A = (b_1 \; b_2 \; b_3 \; b_4)$$

$$e_3 A = (c_1 \; c_2 \; c_3 \; c_4)$$

III

(i) Let $D = (\vec{a}\ \vec{b})$. Then

$$\left(\vec{c} - D\begin{pmatrix} s_0 \\ t_0 \end{pmatrix}, D\begin{pmatrix} s \\ t \end{pmatrix}\right) = 0 \quad \left(\text{for any } \begin{pmatrix} s \\ t \end{pmatrix} \in \mathbb{R}^2\right)$$

$$\Longleftrightarrow \ {}^tDD\begin{pmatrix} s_0 \\ t_0 \end{pmatrix} = {}^tD\vec{c} . \qquad (\#)$$

$${}^tDD = \begin{pmatrix} \|\vec{a}\|^2 & (\vec{a}, \vec{b}) \\ (\vec{b}, \vec{a}) & \|\vec{b}\|^2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 10 \end{pmatrix}$$

$${}^tD\vec{c} = \begin{pmatrix} (\vec{a}, \vec{c}) \\ (\vec{b}, \vec{c}) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Thus

$$(\#) \Longleftrightarrow \begin{pmatrix} 4 & 4 \\ 4 & 10 \end{pmatrix}\begin{pmatrix} s_0 \\ t_0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Accordingly we get

$$\begin{pmatrix} s_0 \\ t_0 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 10 \end{pmatrix}^{-1}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{24}\begin{pmatrix} 10 & -4 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{24}\begin{pmatrix} -14 \\ 8 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} -7 \\ 8 \end{pmatrix}.$$

and find that

$$\|\vec{c} - s\vec{a} - t\vec{b}\|^2$$

has a minimum value

$$\Longleftrightarrow \quad s = -\frac{7}{12}, \ t = \frac{2}{3}$$

(ii)

$$ {}^tDD = \begin{pmatrix} \|\vec{a}\|^2 & (\vec{a}, \vec{b}) \\ (\vec{b}, \vec{a}) & \|\vec{b}\|^2 \end{pmatrix} $$

$$ = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix} $$

$$ {}^tD\vec{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. $$

Thus the normal equation is explicitly written by

$$ {}^tDD \begin{pmatrix} s_0 \\ t_0 \end{pmatrix} = {}^tD\vec{c} \iff \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} s_0 \\ t_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, $$

whose solution is

$$ \begin{pmatrix} s_0 \\ t_0 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} $$

$$ = \frac{1}{11} \begin{pmatrix} 3 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 2 \\ 3 \end{pmatrix}. $$

Accordingly the minimum value of

$$ \|\vec{c} - s\vec{a} - t\vec{b}\|^2 $$

is taken when

$$ s = \frac{2}{11}, \quad t = \frac{3}{11} $$

(iii)

$$ {}^tDD = \begin{pmatrix} \|\vec{a}\|^2 & (\vec{a},\vec{b}) \\ (\vec{b},\vec{a}) & \|\vec{b}\|^2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix} $$

$$ {}^tD\vec{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. $$

Thus the normal equation is

$$ {}^tDD\begin{pmatrix} s_0 \\ t_0 \end{pmatrix} = {}^tD\vec{c} \iff \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} s_0 \\ t_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. $$

whose solution is

$$ \begin{pmatrix} s_0 \\ t_0 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} $$

$$ = \frac{1}{20}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 2 \\ 1 \end{pmatrix} $$

Accordingly the minimum value of

$$ \|\vec{c} - s\vec{a} - t\vec{b}\|^2 $$

is taken when

$$ s = \frac{1}{5}, \quad t = \frac{1}{10}. $$