Matrices and their operations No. 3

Transposition of $m \times n$ Matrices

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Transposition of vectors

Definition

$$
{}^t\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = (a_1 \ldots \ a_n)
$$

$$
{}^t(a_1 \ldots \ a_n) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}
$$

Transposition of vectors – Basic Properties

Basic Properties (1)

- **(i)** $\vec{a},\ \vec{b} \in \mathbf{R}^n$, we have

$$^t\left(\vec{a}+\vec{b}\right) = {}^t\vec{a} + {}^t\vec{b}, \quad {}^t(\lambda\vec{a}) = \lambda\,{}^t\vec{a}$$

- **(ii)** $\mathbf{a},\ \mathbf{b} \in (\mathbf{R}^n)^*$, we have

$$^t\left(\mathbf{a}+\mathbf{b}\right) = {}^t\mathbf{a} + {}^t\mathbf{b}, \quad {}^t(\lambda\mathbf{a}) = \lambda\,{}^t\mathbf{a}$$

Transposition of vectors – Basic Properties

Basic Properties (2)

- **(iii)** $\vec{a}_1, \ldots, \vec{a}_n \in \mathbf{R}^m$, we have

$$ {}^t\left(c_1\vec{a}_1 + \cdots + c_n\vec{a}_n\right) = c_1\,{}^t\vec{a}_1 + \cdots + c_n\,{}^t\vec{a}_n $$

It follows from the basic property (iii) that

$$ {}^t\left((\vec{a}_1 \ \ \ldots \ \ \vec{a}_n)\vec{c}\right) = (c_1 \ \ \ldots \ \ c_n) \begin{pmatrix} {}^t\vec{a}_1 \\ \vdots \\ {}^t\vec{a}_n \end{pmatrix} $$

Transposition of vectors – Basic Properties

The transposition of vectors is related to the dot product by the following basic property (iv).

Basic Properties (3)

- **(iv)** For $\vec{a}$, $\vec{b} \in \mathbf{R}^n$, we have

$$(\vec{a}, \vec{b}) = {}^t\vec{a} \cdot \vec{b}$$

Remark that the both sides are equal to

$$a_1 b_1 + \cdots + a_n b_n$$

Transposition of $m \times n$ Matrices

Definition

$$A = (\vec{a}_1 \ \ldots \ \vec{a}_n) = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} = \begin{pmatrix} a_{11} & \ldots & a_{1j} & \ldots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \ldots & a_{ij} & \ldots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \ldots & a_{mj} & \ldots & a_{mn} \end{pmatrix}$$

Then the transposition of A is a $n \times m$ matrix defined by

$${}^tA = ({}^t\mathbf{a}_1 \ \ldots \ {}^t\mathbf{a}_n) = \begin{pmatrix} {}^t\vec{a}_1 \\ \vdots \\ {}^t\vec{a}_m \end{pmatrix} = \begin{pmatrix} a_{11} & \ldots & a_{i1} & \ldots & a_{m1} \\ \vdots & & \vdots & & \vdots \\ a_{1j} & \ldots & a_{ij} & \ldots & a_{mj} \\ \vdots & & \vdots & & \vdots \\ a_{1n} & \ldots & a_{in} & \ldots & a_{mn} \end{pmatrix}$$

Why do we need such a complicated operation for matrices? We can explain it by the following theorem.

Theorem

Given a $m \times n$ matrix A. Then we have

$$(A\vec{x}, \vec{y}) = (\vec{x}, {}^tA\vec{y})$$

for $\vec{x} \in \mathbf{R}^n$ and $\vec{y} \in \mathbf{R}^m$.

Before giving proof for the theorem, remark that $A\vec{x} \in \mathbf{R}^m$ and that ${}^tA\vec{y} \in \mathbf{R}^n$.

Why we do need the transposition of matrices? (2)

Proof.

$$
\begin{aligned}
LHS &= (x_1\vec{a}_1 + \cdots + x_n\vec{a}_n, \vec{y}) \\
&= x_1(\vec{a}_1, \vec{y}) + \cdots + x_n(\vec{a}_n, \vec{y}) \\
&= \left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} (\vec{a}_1, \vec{y}) \\ \vdots \\ (\vec{a}_n, \vec{y}) \end{pmatrix} \right) = \left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} {}^t\vec{a}_1\vec{y} \\ \vdots \\ {}^t\vec{a}_n\vec{y} \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} {}^t\vec{a}_1 \\ \vdots \\ {}^t\vec{a}_n \end{pmatrix} \vec{y} \right) = (\vec{x}, {}^tA\vec{y}) = RHS
\end{aligned}
$$

□

2 dimensional linear subspaces and a problem

Given $\vec{a}, \vec{b} \in \mathbf{R}^n$ and assume that

$$\vec{a} \nparallel \vec{b}$$

We define the 2-dimensional linear subspace V generated by $\vec{a}$ and $\vec{b}$ by

$$V := \{s_1\vec{a} + s_2\vec{b};\ s_1, s_2 \in \mathbf{R}\}$$

Problem

Given $\vec{c} \in \mathbf{R}^n$. Then consider the problem to find $\vec{v}_0 \in V$ satisfying

$$(\#) \qquad ||\vec{c} - \vec{v}_0||^2 \leq ||\vec{c} - \vec{v}||^2 \quad (\vec{v} \in V)$$

Namely to find $\vec{v}_0 \in V$ minimizing

$$||\vec{c} - \vec{v}||^2 \quad (\vec{v} \in V)$$

Solution to the problem (1)

The first step of the solution is given in the following theorem.

Theorem

If $\vec{v}_o \in V$ satisfies the condition,

$$(\#\#) \qquad (\vec{c} - \vec{v}_0, \vec{v}) = 0 \quad (\vec{v} \in V)$$

then the inequality $(\#)$ holds.

Take any $\vec{v} \in V$. Then

$$\begin{aligned} ||\vec{c} - \vec{v}||^2 &= ||\vec{c} - \vec{v}_0 + (\vec{v}_0 - \vec{v})||^2 \\ &= ||\vec{c} - \vec{v}_0||^2 + ||\vec{v}_0 - \vec{v}||^2 \geq ||\vec{c} - \vec{v}_0||^2 \end{aligned}$$

Remark that the equality holds at the last inequality if

$$||\vec{v}_0 - \vec{v}||^2 = 0 \quad i.e. \quad \vec{v} = \vec{v}_0$$

Solution to the problem (2)

Now we try to find the vector $\vec{v}_0 \in V$ satisfying $(\#\#)$ by using a $n \times 2$ matrix $A = (\vec{a}\ \vec{b})$ and its tranposition ${}^tA$. $\vec{v}_0, \vec{v}$ are expressed with A by

$$\vec{v} = A \begin{pmatrix} s \\ t \end{pmatrix}, \quad \vec{v}_0 = A \begin{pmatrix} s_0 \\ t_0 \end{pmatrix}$$

Then the condition $(\#\#)$ is equivallent to

$$(\vec{c} - A \begin{pmatrix} s_0 \\ t_0 \end{pmatrix}, A \begin{pmatrix} s \\ t \end{pmatrix}) = 0 \quad (\begin{pmatrix} s \\ t \end{pmatrix} \in \mathbf{R}^2)$$

Moreover the condition is equivallent to

$$\left({}^tAA \begin{pmatrix} s_0 \\ t_0 \end{pmatrix} - {}^tA\vec{c}, \begin{pmatrix} s \\ t \end{pmatrix} \right) = 0 \quad (\begin{pmatrix} s \\ t \end{pmatrix} \in \mathbf{R}^2)$$

Solution to the problem (3)

Since $\begin{pmatrix} s \\ t \end{pmatrix} \in \mathbf{R}^2$ is arbitrary, the condition is equivallent to

$$ {}^tAA \begin{pmatrix} s_0 \\ t_0 \end{pmatrix} = {}^tA\vec{c} $$

The matrix ${}^tAA$ is of type 2×2. Moreover we will find that ${}^tAA$ is regular under the condition that $\vec{a} \not\parallel \vec{b}$. Accordingly we have

$$ \vec{v}_0 = \left({}^tAA\right)^{-1} A\vec{c} $$

The Gram Marix

${}^tAA$ is called *Gram matrix* of A. In this case we have

$$ {}^tAA = \begin{pmatrix} {}^t\vec{a} \\ {}^t\vec{b} \end{pmatrix} (\vec{a}\ \vec{b}) = \begin{pmatrix} {}^t\vec{a}\vec{a} & {}^t\vec{a}\vec{b} \\ {}^t\vec{b}\vec{a} & {}^t\vec{b}\vec{b} \end{pmatrix} = \begin{pmatrix} ||\vec{a}||^2 & (\vec{a}, \vec{b}) \\ (\vec{b}, \vec{a}) & ||\vec{b}||^2 \end{pmatrix} $$