## Elementary Matrices (3)

**To multiply jth row by $\lambda \neq 0$**

$$Q_2(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\vec{e}_1\ \lambda\vec{e}_2\ \vec{e}_3) = \begin{pmatrix} \mathbf{e}_1 \\ \lambda\mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

satisfies

$$Q_2(\lambda)\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \lambda\mathbf{b} \\ \mathbf{c} \end{pmatrix}, \quad (\vec{a}\ \vec{b}\ \vec{c})Q_2(\lambda) = (\vec{a}\ \lambda\vec{b}\ \vec{c})$$

$$Q_1(\lambda) = \begin{pmatrix} \lambda & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad Q_1(\lambda)\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \lambda\mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix}$$



$$Q_3(\lambda) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \lambda \end{pmatrix} \quad Q_3(\lambda)\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \lambda\mathbf{c} \end{pmatrix}$$

## Elementary Matrices (3)

**To multiply jth row by $\lambda \neq 0$**

$Q_i(\lambda)$ $(i = 1, 2, 3)$ are regular.

In fact we have

$$Q_2(\lambda)Q_2(\mu) = Q_2(\lambda\mu), \quad Q_2(1) = I_3$$

$$\begin{pmatrix} 1 & & \\ & \mu & \\ & & 1 \end{pmatrix} \xrightarrow{Q_2(\lambda)} \begin{pmatrix} 1 & & \\ & \lambda\mu & \\ & & 1 \end{pmatrix} = Q_2(\lambda\mu)$$

Thus if $\lambda \neq 0$

$$Q_2(\lambda)Q_2(\frac{1}{\lambda}) = Q_2(\frac{1}{\lambda})Q_2(\lambda) = Q_2(1) = I_3$$

$Q_2(\lambda)$ is regular and

$$Q_2(\lambda)^{-1} = Q_2(\tfrac{1}{\lambda}).$$

## Scalar Multiplication to Matirces

### Scalar Multiplication to $m \times n$ Matrices

Given a $m \times n$ matix

$$A = (\vec{a}_1 \ \dots \ \vec{a}_n) = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix},$$

we define a scalar multiplication by $\lambda$ to $A$ as follows:

$$\lambda A = (\lambda\vec{a}_1 \ \dots \ \lambda\vec{a}_n) = \begin{pmatrix} \lambda\mathbf{a}_1 \\ \vdots \\ \lambda\mathbf{a}_m \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \vdots & & \vdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{pmatrix}$$



## Scalar Multiplication to Matirces

$\lambda(\mu\vec{x}) = (\lambda\mu)\vec{x} = \mu(\lambda\vec{x})$

### Theorem

- **(i)** $(\lambda A)\vec{x} = \lambda(A\vec{x}) = A(\lambda\vec{x})$
- **(ii)** $(\lambda A)X = \lambda(AX) = A(\lambda X)$

The proof for **(i)** is given as follows:

$$\begin{aligned} (\lambda A)\vec{x} &= (\lambda\vec{a}_1 \ \dots \ \lambda\vec{a}_n)\vec{x} \\ &= x_1(\lambda\vec{a}_1) + \cdots + x_n(\lambda\vec{a}_n) = (*) \\ &= \lambda(x_1\vec{a}_1) + \cdots + \lambda(x_n\vec{a}_n) \\ &= \lambda(x_1\vec{a}_1 + \cdots + x_n\vec{a}_n) = \lambda(A\vec{x}) \\ (*) &= (\lambda x_1)\vec{a}_1 + \cdots + (\lambda x_n)\vec{a}_n = A(\lambda\vec{x}) \end{aligned}$$

$n$

$= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$(\vec{a}_1 \ \cdots \ \vec{a}_n)\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix}$

Moreover the property **(ii)** is derived easily from **(i)**.



$$\begin{aligned} (\lambda A)X &= (\lambda A)(\vec{x}_1 \ \cdots \ \vec{x}_\ell) \\ &= ((\lambda A)\vec{x}_1 \ \cdots \ (\lambda A)\vec{x}_\ell) \\ &= (\lambda(A\vec{x}_1) \ \cdots \ \lambda(A\vec{x}_\ell)) = \lambda(A\vec{x}_1 \ \cdots \ A\vec{x}_\ell) \end{aligned}$$

$\lambda(AX)$

$\parallel$

## Other Basic Properties of Scalar Multiplication

**Theorem**

- **(iii)** $(\lambda+\mu)A = \lambda A + \mu A$
- **(iv)** $(\lambda\mu)A = \lambda(\mu A)$
- **(v)** $1A = A$ and $0A = O_{m,n}$

$\} \leftarrow$ Prove these.

These properties can derived from the following corresponding properties for vectors. It is necessary to define the addition of matrices to understand (iii), and we put it off for the moment.

- **(iii)** $(\lambda+\mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$
- **(iv)** $(\lambda\mu)\vec{a} = \lambda(\mu\vec{a})$
- **(v)** $1\vec{a} = \vec{a}$ and $0\vec{a} = \vec{0}$



$$(\lambda+\mu)(\vec{a}_1 \cdots \vec{a}_n) = ((\lambda+\mu)\vec{a}_1 \cdots (\lambda+\mu)\vec{a}_n)$$
$$= (\lambda\vec{a}_1 + \mu\vec{a}_1 \cdots \lambda\vec{a}_n + \mu\vec{a}_n)$$
$$= (\lambda\vec{a}_1 \cdots \lambda\vec{a}_n) + (\mu\vec{a}_1 \cdots \mu\vec{a}_n)$$
$$= \lambda A + \mu A$$

## Addition of two $m \times n$ Matrices

**Definition**

Given two $m \times n$ matrices

$$A = (\vec{a}_1 \ \dots \ \vec{a}_n) = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{im} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

$$B = (\vec{b}_1 \ \dots \ \vec{b}_n) = \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_m \end{pmatrix} = \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ \vdots & & \vdots & & \vdots \\ b_{i1} & \cdots & b_{ij} & \cdots & b_{im} \\ \vdots & & \vdots & & \vdots \\ b_{m1} & \cdots & b_{mj} & \cdots & b_{mn} \end{pmatrix}$$

## Addition of two $m \times n$ Matrices

### Definition

Then

$$A + B = (\vec{a}_1 + \vec{b}_1 \ \dots \ \vec{a}_n + \vec{b}_n) = \begin{pmatrix} \mathbf{a}_1 + \mathbf{b}_1 \\ \vdots \\ \mathbf{a}_m + \mathbf{b}_m \end{pmatrix}$$

$$= \begin{pmatrix} & \vdots & \\ \cdots & a_{ij} + b_{ij} & \cdots \\ & \vdots & \end{pmatrix} \leftarrow i\text{th row}$$

$j$th column



## Basic Properties (1)

### Basic Properties (1)

- **(i)** $(A + B) + C = A + (B + C)$
- **(ii)** $A + O_{m,n} = O_{m,n} + A = A$
- **(iii)** $A + B = B + A$
- **(iv)** $\lambda(A + B) = \lambda A + \lambda B$
- **(v)** $(\lambda + \mu)A = \lambda A + \mu A$

Prove these by yourself.

(i) follows from $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.
(ii) follows from $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$.
(iii) follows from $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.
(v) follows from $(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$. (iv) is proved as follows.

$$LHS = \lambda(\dots \vec{a}_j + \vec{b}_j \dots) = (\dots \lambda(\vec{a}_j + \vec{b}_j) \dots)$$
$$= (\dots \lambda\vec{a}_j + \lambda\vec{b}_j \dots) = (\dots \lambda\vec{a}_j \dots) + (\dots \lambda\vec{b}_j \dots) = RHS$$



(i) $((\cdots \vec{a}_j \cdots) + (\cdots \vec{e}_j \cdots)) + (\cdots \vec{c}_j \cdots)$

$= (\cdots \vec{a}_j + \vec{e}_j \cdots) + (\cdots \vec{c}_j \cdots)$

$= (\cdots (\vec{a}_j + \vec{e}_j) + \vec{c}_j \cdots)$

(ii) $(\cdots \vec{a}_j \cdots) + (\cdots \vec{0} \cdots) = (\cdots \vec{a}_j + \vec{0} \cdots)$

$\qquad\qquad\qquad\qquad\quad \uparrow$

$\qquad\qquad\qquad\quad j\text{th}$

$= (\cdots \vec{a}_j \quad )$

$= A$

$A: m \times \underline{n} \quad X \quad \underline{n} \times \ell \longrightarrow AX \quad m \times \ell.$

## Basic Properties (2)

### Basic Properties (2)

- **(vi)** For $n \times \ell$ matrices $X$ and $Y$, we have

$$\underline{A(X+Y)} = AX + AY$$

- **(vii)** For $s \times n$ matrices $P$ and $Q$, we have

$m$

$$(P+Q)A = PA + QA$$

$P \quad s \times \underline{m} \quad A: \underline{m} \times n$

$\longrightarrow PA \quad s \times n.$

(vi) follows from $\underline{A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}}$. In fact

$$A(X+Y) = LHS = A(\ldots \vec{x}_k + \vec{y}_k \ldots) = (\ldots A(\vec{x}_k + \vec{y}_k) \ldots)$$
$$= (\ldots A\vec{x}_k + A\vec{y}_k \ldots) = \underset{\substack{\parallel \\ AX}}{(\ldots A\vec{x}_k \ldots)} + \underset{\substack{\parallel \\ AY.}}{(\ldots A\vec{y}_k \ldots)} = RHS$$



## Basic Properties (2)

(vii) follows from the identity $(P+Q)\vec{a} = P\vec{a} + Q\vec{a}$ which is derived in the following way.

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

$$(P+Q)\vec{a} = LHS = (\vec{p}_1 + \vec{q}_1 \cdots \vec{p}_m + \vec{q}_m)\vec{a}$$
$$= a_1(\vec{p}_1 + \vec{q}_1) + \cdots + a_m(\vec{p}_m + \vec{q}_m)$$
$$= \cdots = \underbrace{(a_1\vec{p}_1 + \cdots + a_m\vec{p}_m)}_{\substack{\parallel \\ P\vec{a}}} + \underbrace{(a_1\vec{q}_1 + \cdots + a_m\vec{q}_m)}_{\substack{\parallel \\ Q\vec{a}}}$$
$$= RHS$$

$$= a_1\vec{p}_1 + a_1\vec{q}_1 + \cdots + a_m\vec{p}_m + a_m\vec{q}_m$$



$$(P+Q)(\vec{a}_1 \cdots \vec{a}_n) = ((P+Q)\vec{a}_1 \cdots (P+Q)\vec{a}_n)$$
$$= (P\vec{a}_1 + Q\vec{a}_1 \quad \cdots \quad P\vec{a}_n + Q\vec{a}_n)$$
$$= (P\vec{a}_1 \cdots P\vec{a}_n) + (Q\vec{a}_1 \cdots Q\vec{a}_n) = PA + QA$$

# Matrices and their operations No. 3

## Transposition of $m \times n$ Matrices

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## Transposition of vectors

**Definition**

$$
{}^t\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = (a_1 \ldots\ a_n)
$$

$$
{}^t(a_1 \ldots\ a_n) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}
$$

## Transposition of vectors – Basic Properties

### Basic Properties (1)

- **(i)** $\vec{a},\ \vec{b} \in \mathbf{R}^n$, we have

$$ {}^t\left(\vec{a}+\vec{b}\right) = {}^t\vec{a} + {}^t\vec{b}, \quad {}^t(\lambda\vec{a}) = \lambda\,{}^t\vec{a} $$

- **(ii)** $\mathbf{a},\ \mathbf{b} \in (\mathbf{R}^n)^*$, we have

$$ {}^t(\mathbf{a}+\mathbf{b}) = {}^t\mathbf{a} + {}^t\mathbf{b}, \quad {}^t(\lambda\mathbf{a}) = \lambda\,{}^t\mathbf{a} $$

prove this.



in Two weeks.

## Transposition of vectors – Basic Properties

### Basic Properties (2)

- **(iii)** $\vec{a}_1, \ldots, \vec{a}_n \in \mathbf{R}^m$, we have

$$ {}^t(c_1\vec{a}_1 + \cdots + c_n\vec{a}_n) = c_1\,{}^t\vec{a}_1 + \cdots + c_n\,{}^t\vec{a}_n $$

It follows from the basic property (iii) that

$$ {}^t((\vec{a}_1 \ \ldots \ \vec{a}_n)\vec{c}) = (c_1 \ \ldots \ c_n) \begin{pmatrix} {}^t\vec{a}_1 \\ \vdots \\ {}^t\vec{a}_n \end{pmatrix} $$

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$t\,(\vec{a} + \vec{b}) = t \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

$$= (a_1 + b_1 \quad a_2 + b_2 \quad a_3 + b_3)$$

$$= (a_1 \quad a_2 \quad a_3) + (b_1 \quad b_2 \quad b_3)$$

$$= t\,\vec{a} + t\,\vec{b}$$

## Transposition of vectors – Basic Properties

The transposition of vectors is related to the dot product by the following basic property (iv).

### Basic Properties (3)

- **(iv)** For $\vec{a},\ \vec{b} \in \mathbf{R}^n$, we have

$$(\vec{a}, \vec{b}) = {}^t\vec{a} \cdot \vec{b}$$

Remark that the both sides are equal to

$$a_1 b_1 + \cdots + a_n b_n$$

$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$



$(\vec{a}, \vec{b}) = a_1 b_1 + a_2 b_2 + a_3 b_3$

${}^t\vec{a}\,\vec{b} = (a_1\ a_2\ a_3) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

## Transposition of $m \times n$ Matrices

### Definition

$$A = (\vec{a}_1 \ \dots \ \vec{a}_n) = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

$\mathbf{a}_i$

Then the transposition of $A$ is a $n \times m$ matrix defined by

$${}^tA = ({}^t\mathbf{a}_1 \ \dots {}^t\mathbf{a}_n) = \begin{pmatrix} {}^t\vec{a}_1 \\ \vdots \\ {}^t\vec{a}_m \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{i1} & \dots & a_{m1} \\ \vdots & & \vdots & & \vdots \\ a_{1j} & \dots & a_{ij} & \dots & a_{mj} \\ \vdots & & \vdots & & \vdots \\ a_{1n} & \dots & a_{in} & \dots & a_{mn} \end{pmatrix}$$

$= {}^t\vec{a}_j$



${}^t\mathbf{a}_i$

## Why do we need the transposition of matrices?

Why do we need such a complicated operation for matrices? We can explain it by the following theorem.

**Theorem**

Given a $m \times n$ matrix $A$. Then we have

$$(A\vec{x}, \vec{y}) = (\vec{x}, {}^tA\vec{y})$$

for $\vec{x} \in \mathbf{R}^n$ and $\vec{y} \in \mathbf{R}^m$.

Before giving proof for the theorem, remark that $A\vec{x} \in \mathbf{R}^m$ and that ${}^tA\vec{y} \in \mathbf{R}^n$.

$\parallel$

$x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$

${}^tA : n \times m$



$= ({}^t\vec{a1}_1 \cdots {}^ta1_m)$

${}^tA\vec{y} \in \mathbb{R}^n$

$= y_1\,{}^ta1_1 + \cdots + y_m\,{}^ta1_m$

## Why we do need the transposition of matrices? (2)

$= A\vec{x}$

**Proof.**

$(\vec{a}, \vec{b}) = {}^t\vec{a}\,\vec{b}$

$$\begin{aligned}
LHS &= (\boxed{x_1\vec{a}_1 + \cdots + x_n\vec{a}_n}, \vec{y}) \\
&= x_1(\vec{a}_1, \vec{y}) + \cdots + x_n(\vec{a}_n, \vec{y}) \\
&= \left( \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} (\vec{a}_1, \vec{y}) \\ \vdots \\ (\vec{a}_n, \vec{y}) \end{pmatrix} \right) = \left( \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} {}^t\vec{a}_1\vec{y} \\ \vdots \\ {}^t\vec{a}_n\vec{y} \end{pmatrix} \right) \\
&= \left( \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} {}^t\vec{a}_1 \\ \vdots \\ {}^t\vec{a}_n \end{pmatrix} \vec{y} \right) = (\vec{x}, {}^tA\vec{y}) = RHS
\end{aligned}$$

□

## 2 dimensional linear subspaces and a problem

Given $\vec{a}, \vec{b} \in \mathbf{R}^n$ and assume that

$$\vec{a} \nparallel \vec{b}$$

We define the 2-dimensional linear subspace $V$ generated by $\vec{a}$ and $\vec{b}$ by

$$V := \{s_1\vec{a} + s_2\vec{b};\ \ s_1, s_2 \in \mathbf{R}\}$$

**Problem**

Given $\vec{c} \in \mathbf{R}^n$. Then consider the problem to find $\vec{v}_0 \in V$ satisfying

$$(\#) \qquad ||\vec{c} - \vec{v}_0||^2 \leq ||\vec{c} - \vec{v}||^2 \quad (\vec{v} \in V)$$

Namely to find $\vec{v}_0 \in V$ minimizing

$$||\vec{c} - \vec{v}||^2 \quad (\vec{v} \in V)$$



$\vec{v}_1, \vec{v}_2 \in V \Rightarrow \vec{v}_1 + \vec{v}_2 \in V$

## Solution to the problem (1)

The first step of the solution is given in the following theorem.

**Theorem**

If $\vec{v}_o \in V$ satisfies the condition,

for all

$$(\#\#) \qquad (\vec{c} - \vec{v}_0, \vec{v}) = 0 \quad (\vec{v} \in V)$$

then the inequality $(\#)$ holds. $\Leftrightarrow \vec{c} - \vec{v}_0 \perp V$.

Take any $\vec{v} \in V$. Then

$\vec{v}_0 - \vec{v} \in V$

$\vec{c} - \vec{v}_0 \perp \vec{v}_0 - \vec{v}$

$$\begin{aligned} ||\vec{c} - \vec{v}||^2 &= ||\vec{c} - \vec{v}_0 + (\vec{v}_0 - \vec{v})||^2 \\ &= ||\vec{c} - \vec{v}_0||^2 + \underbrace{||\vec{v}_0 - \vec{v}||^2}_{\geq 0} \geq ||\vec{c} - \vec{v}_0||^2 \end{aligned}$$

Remark that the equality holds at the last inequality if

$$||\vec{v}_0 - \vec{v}||^2 = 0 \quad i.e. \quad \vec{v} = \vec{v}_0$$

## Solution to the problem (2)

Now we try to find the vector $\vec{v}_0 \in V$ satisfying (##) by using a $n \times 2$ matrix $A = (\vec{a}\ \vec{b})$ and its tranposition ${}^tA$. $\vec{v}_0, \vec{v}$ are expressed with $A$ by

$$\vec{v} = A\begin{pmatrix} s \\ t \end{pmatrix}, \quad \vec{v}_0 = A\begin{pmatrix} s_0 \\ t_0 \end{pmatrix}$$

$s_0\vec{a} + t_0\vec{b} \in V$

Then the condition (##) is equivallent to

$= s\vec{a} + t\vec{b} \in V$

$$(\vec{c} - A\begin{pmatrix} s_0 \\ t_0 \end{pmatrix}, A\begin{pmatrix} s \\ t \end{pmatrix}) = 0 \quad (\begin{pmatrix} s \\ t \end{pmatrix} \in \mathbf{R}^2)$$

Moreover the condition is equivallent to

$$\left(\boxed{{}^tAA\begin{pmatrix} s_0 \\ t_0 \end{pmatrix} - {}^tA\vec{c}}, \begin{pmatrix} s \\ t \end{pmatrix}\right) = 0 \quad (\begin{pmatrix} s \\ t \end{pmatrix} \in \mathbf{R}^2)$$



Prop. $(\vec{a}, \vec{v}) = 0$ for any $\vec{v} \in \mathbb{R}^n$
$\vec{a} \in \mathbb{R}^n$ $\Rightarrow \vec{a} = \vec{0}$

## Solution to the problem (3)

Since $\begin{pmatrix} s \\ t \end{pmatrix} \in \mathbf{R}^2$ is arbitrary, the condition is equivallent to

$${}^tAA\begin{pmatrix} s_0 \\ t_0 \end{pmatrix} = {}^tA\vec{c}$$

The matrix ${}^tAA$ is of type $2 \times 2$. Moreover we will find that ${}^tAA$ is regular under the condition that $\vec{a} \nparallel \vec{b}$. Accordingly we have

$$\vec{v}_0 = ({}^tAA)^{-1} A\vec{c}$$

### The Gram Marix

${}^tAA$ is called *Gram matrix* of $A$. In this case we have

$${}^tAA = \begin{pmatrix} {}^t\vec{a} \\ {}^t\vec{b} \end{pmatrix} (\vec{a}\ \vec{b}) = \begin{pmatrix} {}^t\vec{a}\vec{a} & {}^t\vec{a}\vec{b} \\ {}^t\vec{b}\vec{a} & {}^t\vec{b}\vec{b} \end{pmatrix} = \begin{pmatrix} ||\vec{a}||^2 & (\vec{a}, \vec{b}) \\ (\vec{b}, \vec{a}) & ||\vec{b}||^2 \end{pmatrix}$$

$$\vec{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A = (\vec{a} \ \vec{b})$$

$$ {}^tA A = \begin{pmatrix} \|\vec{a}\|^2 & (\vec{a}, \vec{b}) \\ (\vec{b}, \vec{a}) & \|\vec{b}\|^2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$$

$$ {}^tA \vec{c} = \begin{pmatrix} {}^t\vec{a} \\ {}^t\vec{b} \end{pmatrix} \vec{c} = \begin{pmatrix} (\vec{a}, \vec{c}) \\ (\vec{b}, \vec{c}) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} s_0 \\ t_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} = 2 \cdot 3 - 2 \cdot 2 = 2$$

$$\begin{pmatrix} s_0 \\ t_0 \end{pmatrix} = \overbrace{\frac{1}{2} \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}}^{= \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}^{-1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The minimum value is attained

at $(s, t) = \left(\frac{1}{2}, 0\right)$