Exercise for TA session 2016/11/01

I Given 3×3 matrices A, B, C and Assume that they are regular.

(i) show that AB is regular.

(ii) show That ABC is regular.

(iii) Show that A^{-1} is regular.

II Find a 4×4 matrix P satisfying the given identity

(i) $$P\begin{pmatrix} \mathbb{a} \\ \mathbb{b} \\ \mathbb{c} \\ \mathbb{d} \end{pmatrix} = \begin{pmatrix} \mathbb{c} \\ \mathbb{b} \\ \mathbb{a} \\ \mathbb{d} \end{pmatrix}$$

(ii) $$P\begin{pmatrix} \mathbb{a} \\ \mathbb{b} \\ \mathbb{c} \\ \mathbb{d} \end{pmatrix} = \begin{pmatrix} \mathbb{d} \\ \mathbb{b} \\ \mathbb{c} \\ \mathbb{a} \end{pmatrix}$$

(iii) $$P\begin{pmatrix} \mathbb{a} \\ \mathbb{b} \\ \mathbb{c} \\ \mathbb{d} \end{pmatrix} = \begin{pmatrix} \mathbb{a} \\ \mathbb{b} \\ \lambda \mathbb{a} + \mathbb{c} \\ \mathbb{d} \end{pmatrix}$$

(iv) $$P\begin{pmatrix} \mathbb{a} \\ \mathbb{b} \\ \mathbb{c} \\ \mathbb{d} \end{pmatrix} = \begin{pmatrix} \mathbb{a} \\ \mathbb{b} \\ \mathbb{c} \\ \lambda \mathbb{b} + \mathbb{d} \end{pmatrix}$$

(v) $$P\begin{pmatrix} \mathbb{a} \\ \mathbb{b} \\ \mathbb{c} \\ \mathbb{d} \end{pmatrix} = \begin{pmatrix} \mathbb{a} \\ \mathbb{b} \\ \lambda \mathbb{c} \\ \mathbb{d} \end{pmatrix}$$

III Let $A = \begin{pmatrix} a & p & q \\ 0 & b & r \\ 0 & 0 & c \end{pmatrix}$ and assume that $abc \neq 0$.

Show that A is regular and find the inverse of A.

IV $\vec{a}_1, \dots, \vec{a}_\ell \in \mathbb{R}^n$ and a regular $n \times n$ matrix P is given. We define the vectors b_j by

$$\vec{b}_1 = P\vec{a}_1, \dots, \vec{b}_j = P\vec{a}_j, \dots, \vec{b}_\ell = P\vec{a}_\ell .$$

Show that $\vec{a}_1, \dots, \vec{a}_\ell$ are linearly independent iff $\vec{b}_1, \dots, \vec{b}_\ell$ are linearly independent.

I (i)

$$B^{-1}A^{-1}AB = B^{-1}I_3B = B^{-1}B = I_3$$

$$ABB^{-1}A^{-1} = AI_3A^{-1} = AA^{-1} = I_3$$

Accordingly AB is regular and

$$(AB)^{-1} = B^{-1}A^{-1}$$

(ii) By (i) we find AB is regular. We employ (i) again to find $(AB)C$ is regular.

To find $(ABC)^{-1}$ we use (i) twice.

$$((AB)C)^{-1} = C^{-1}(AB)^{-1} = C^{-1}B^{-1}A^{-1}$$

(iii) $AA^{-1} = A^{-1}A = I_3$.

Thus A^{-1} is regular and $(A^{-1})^{-1} = A$

II (i) $P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(ii) $P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

(iii) $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \lambda & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(iv) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \lambda & 0 & 1 \end{pmatrix}$

(v) $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

III $\quad X = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$ we assume $AX = XA = I_3$.

$$AX = \begin{pmatrix} a x_1 & * & * \\ a x_2 & * & * \\ a x_3 & a y_3 & * \end{pmatrix} = I_3 .$$

Thus we get $x_2 = x_3 = y_3 = 0$. We calculate again AX:

$$AX = \begin{pmatrix} x_1 & y_1 & z_1 \\ 0 & y_2 & z_2 \\ 0 & 0 & z_3 \end{pmatrix} \begin{pmatrix} a & p & q \\ 0 & b & r \\ 0 & 0 & c \end{pmatrix}$$

$$= \begin{pmatrix} a x_1 & p x_1 + b y_1 & x_1 q + y_1 r + z_1 c \\ 0 & b y_2 & y_2 r + z_2 c \\ 0 & 0 & c z_3 \end{pmatrix} = I_3$$

We get $a x_1 = 1$, $b y_2 = 1$, $c z_3 = 1$. Thus

$$x_1 = \frac{1}{a}, \quad y_2 = \frac{1}{b}, \quad z_3 = \frac{1}{c},$$

what is more we get

$$p x_1 + b y_1 = 0, \quad x_1 q + y_1 r + z_1 c = 0, \quad y_2 r + z_2 c = 0.$$

Thus

$$y_1 = -\frac{p}{b} x_1 = -\frac{p}{ab}, \quad z_1 = -\frac{q}{ac} - \frac{r}{bc}$$

$$z_2 = -\frac{r}{bc}.$$

We calculate

$$XA = \begin{pmatrix} \frac{1}{a} & -\frac{p}{ab} & -\frac{qb + ra}{abc} \\ 0 & \frac{1}{b} & -\frac{r}{bc} \\ 0 & 0 & 1/c \end{pmatrix} \begin{pmatrix} a & p & q \\ 0 & b & r \\ 0 & 0 & c \end{pmatrix} = I_3 .$$

IV

($\Rightarrow$) We assume that $\vec{a}_1, \dots, \vec{a}_l$ are linearly independent. We assume

$$c_1 \vec{e}_1 + \dots + c_l \vec{e}_l = \vec{0}$$

and multiply P^{-1} to the both side to get

$$c_1 P^{-1} \vec{e}_1 + \dots + c_l P^{-1} \vec{e}_l = \vec{0}.$$

Since $P^{-1}\vec{e}_j = a_j$, we get

$$c_1 \vec{a}_1 + \dots + c_l \vec{a}_l = \vec{0}.$$

Accordingly $c_1 = c_2 = \dots = c_l = 0$ since $\vec{a}_1, \dots, \vec{a}_l$ are linearly independent.

($\Leftarrow$) We assume that $\vec{e}_1, \dots, \vec{e}_l$ are linearly independent. We assume

$$c_1 \vec{a}_1 + \dots + c_l \vec{a}_l = \vec{0}$$

and multiply P to get

$$c_1 P \vec{a}_1 + \dots + c_l P \vec{a}_l = \vec{0}$$

This means

$$c_1 \vec{e}_1 + \dots + c_l \vec{e}_l = 0.$$

Since $\vec{e}_1, \dots, \vec{e}_l$ are linearly independent, we get $c_1 = \dots = c_l = 0$.