Linearity of F_A

Linearity of F_A

F_A satisfies the following basic properties called **Linearity**.

- **(i)** $F_A(\vec{x}+\vec{y}) = F_A(\vec{x}) + F_A(\vec{y})$
- **(ii)** $F_A(\lambda\vec{x}) = \lambda F_A(\vec{x})$
- **(iii)** $F_A(\lambda\vec{x}+\mu\vec{y}) = \lambda F_A(\vec{x}) + \mu F_A(\vec{y})$

These three propeties are identical to the following.

- **(i)** $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$
- **(ii)** $A(\lambda\vec{x}) = \lambda(A\vec{x})$
- **(iii)** $A(\lambda\vec{x}+\mu\vec{y}) = \lambda(A\vec{x}) + \mu(A\vec{y})$

Moreover remark that **(iii)** can be easily derived from **(i)** and **(ii)**.

$$A(\lambda\vec{x}+\mu\vec{y}) = A(\lambda\vec{x}) + A(\mu\vec{y}) = \lambda(A\vec{x}) + \mu(A\vec{y})$$

(i) (ii)

$A(\vec{x}+\vec{y})$

$(\vec{a}_1 \cdots \vec{a}_n)$

Proof

Proof for (i)

$$LHS = A\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\right) = A\begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix}$$
$$= (x_1+y_1)\vec{a}_1 + \cdots + (x_n+y_n)\vec{a}_n$$
$$= x_1\vec{a}_1 + y_1\vec{a}_1 + \cdots + x_n\vec{a}_n + y_n\vec{a}_n$$
$$= (x_1\vec{a}_1 + \cdots + x_n\vec{a}_n) + (y_1\vec{a}_1 + \cdots + y_n\vec{a}_n)$$
$$= A\vec{x} + A\vec{y} = RHS$$

Proof for (ii)

$$LHS = A\begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix} = (\lambda x_1)\vec{a}_1 + \cdots + (\lambda x_n)\vec{a}_n$$
$$= \lambda(x_1\vec{a}_1) + \cdots + \lambda(x_n\vec{a}_n) = \lambda(x_1\vec{a}_1 + \cdots + x_n\vec{a}_n) = RHS$$

Corollary

Corollary

n

Let A be a $m \times n$ matrix, $\vec{x}_1, \ldots, \vec{x}_\ell \in \mathbf{R}^\ell$. Then we have

- **(i)** $A(\vec{x}_1 + \cdots + \vec{x}_\ell) = A\vec{x}_1 + \cdots + A\vec{x}_\ell$
- **(ii)** $A(c_1\vec{x}_1 + \cdots + c_\ell\vec{x}_\ell) = c_1(A\vec{x}_1) + \cdots + c_\ell(A\vec{x}_\ell)$

(i) follows from linearity with the aid of induction on ℓ.

$$= A(c_1\vec{x}_1) + \cdots + A(c_\ell\vec{x}_\ell)$$

$$= c_1(A\vec{x}_1) + \cdots + c_\ell(A\vec{x}_\ell)$$

$$A\left((\vec{x}_1 \cdots \vec{x}_\ell)\begin{pmatrix} c_1 \\ \vdots \\ c_\ell \end{pmatrix}\right) = (A\vec{x}_1 \cdots A\vec{x}_\ell)\begin{pmatrix} c_1 \\ \vdots \\ c_\ell \end{pmatrix}$$

$X = (\vec{x}_1 \cdots \vec{x}_\ell)$ $\quad A(X\vec{c}) = (AX)\vec{c}$

$$AX\vec{c}$$

Associativity (1)

Thanks to the Linearity, we can prove the following theorem about **Associativity**.

Theorem: Associativity

Given a $m \times n$ matrix A and a $n \times \ell$ matrix X. Then we have for $\vec{c} \in \mathbf{R}^\ell$.

$$(AX)\vec{c} = A(X\vec{c})$$

In fact,

$$\begin{aligned} RHS &= A(c_1\vec{x}_1 + \cdots + c_n\vec{x}_n) \\ &= c_1(A\vec{x}_1) + \cdots + c_n(A\vec{x}_n) \\ &= (A\vec{x}_1 \ \ldots \ A\vec{x}_n)\vec{c} = (AX)\vec{c} = LHS \end{aligned}$$

$m \times n$ matrix $A = (\vec{a}_1 \cdots \vec{a}_n) = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$

Multiplication: Matrix $\times$ Matrix

Multiplication: Matrix $\times$ Matrix

Take another matrix of type $n \times \ell$: n row and ℓ colums.

$$X = (\vec{x}_1 \ \ldots \ \vec{x}_\ell) = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

Then a $m \times \ell$ matrix AX is defined by

$$AX = (A\vec{x}_1 \ \ldots \ A\vec{x}_\ell) = \begin{pmatrix} \mathbf{a}_1\vec{x}_1 & \ldots & \mathbf{a}_1\vec{x}_\ell \\ \vdots & & \vdots \\ \mathbf{a}_m\vec{x}_1 & \ldots & \mathbf{a}_m\vec{x}_\ell \end{pmatrix}$$

$a_i \vec{x}_j$

ith row and jth column

In case $A = a_1$ $\qquad a_1 X = (a_1 \vec{x}_1 \cdots a_1 \vec{x}_\ell)$

Linear Map defined by A

Linear Map defined by A

We can define a map

$$\begin{aligned} F_A : \mathbf{R}^n &\longrightarrow \mathbf{R}^m \\ \vec{x} &\mapsto A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n \end{aligned}$$

$$a_1 x_1 + \cdots + a_j x_j + \cdots + a_n x_n$$

$$= a_1 (x_{11} \;\cdots\; x_{1i} \;\cdots\; x_{1l})$$

$$+ \;\cdots$$

$$+ a_j (x_{j1} \;\cdots\; x_{ji} \;\cdots\; x_{jl})$$

$$+ \;\cdots$$

$$+ a_n (x_{n1} \;\cdots\; x_{ni} \;\cdots\; x_{nl})$$

ith component

$$= (\;\cdots\; a_1 x_{1i} + \cdots + a_j x_{ji} + \cdots + a_n x_{ni} \;\cdots\;)$$

$$= (\;\cdots\; \overset{(a_1 \cdots a_n) =}{\mathrm{al}}\, \vec{x}_i \;\cdots\;) \qquad \vec{x}_i = \begin{pmatrix} x_{1i} \\ \vdots \\ x_{ji} \\ \vdots \\ x_{ni} \end{pmatrix}$$

$$= \mathrm{al}\, (\vec{x}_1 \cdots \vec{x}_i \cdots \vec{x}_l)$$

$$= \mathrm{al}\, X.$$

$$\boxed{\begin{aligned} \mathrm{al}\,X &= (\mathrm{al}\,\vec{x}_1 \;\cdots\; \mathrm{al}\,\vec{x}_l) \\ &= a_1 x_1 + \cdots + a_n x_n \end{aligned}}$$

Added after class.

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} = (\vec{x}_1 \cdots \vec{x}_l)$$

$$a_1 x_1 + \cdots + a_i x_i + \cdots + a_n x_n$$

$$= a_1 (x_{11} \cdots x_{1j} \cdots x_{1l})$$

$$+ \;\cdots$$

$$+ a_i (x_{i1} \cdots x_{ij} \cdots x_{il})$$

$$+ \;\cdots$$

$$+ a_n (x_{n1} \cdots x_{nj} \cdots x_{nl})$$

$$= (\;\cdots\; \underbrace{\boxed{a_1 x_{1j} + \cdots + a_i x_{ij} + \cdots + a_n x_{nj}}}_{j\text{th component}} \;\cdots\;)$$

$$= (\;\cdots\; \mathrm{al}\,\vec{x}_j \;\cdots\;) = \mathrm{al}\,X$$

$$(1 \;\; 0 \;\; 0) \begin{pmatrix} \rangle\phi_1 \\ \rangle\phi_2 \\ \rangle\phi_3 \end{pmatrix} = \rangle\phi_1$$

$$(\alpha \;\; \beta \;\; \gamma) \begin{pmatrix} \rangle\phi_1 \\ \rangle\phi_2 \\ \rangle\phi_3 \end{pmatrix} = \alpha \rangle\phi_1 + \beta \rangle\phi_2 + \gamma \rangle\phi_3 .$$

Multiplication of matrices expressed by rows

On the other hand

$$
\begin{aligned}
&a_1\mathbf{x}_1 + \cdots + a_j\mathbf{x}_j + \cdots + a_n\mathbf{x}_n \\
=&a_1(x_{11} \ \dots \ x_{1j} \ \dots \ x_{1\ell}) \\
&\quad \vdots \\
&+ a_i(x_{i1} \ \dots \ x_{ij} \ \dots \ x_{i\ell}) \\
&\quad \vdots \\
&+ a_n(x_{n1} \ \dots \ x_{nj} \ \dots \ x_{n\ell}) \\
=&(\dots \ a_1x_{1j} + \cdots + a_ix_{ij} + \cdots + a_nx_{n\ell} \ \dots) \\
=&(\mathbf{a}\vec{x}_1 \ \dots \ \mathbf{a}\vec{x}_j \ \dots \ \mathbf{a}\vec{x}_\ell) \\
=&\mathbf{a}X
\end{aligned}
$$

i $\quad i$ $\quad ij$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix}$$

Multiplication of matrices expressed by rows

In case A is a row vector

For a n dim. row vector $\mathbf{a} = (a_1 \ \dots \ a_n)$ and a $n \times \ell$ matrix

$$X = (\vec{x}_1 \ \dots \ \vec{x}_\ell) = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

we have

$$
\begin{aligned}
\mathbf{a}X &= (\mathbf{a}\vec{x}_1 \ \dots \ \mathbf{a}\vec{x}_\ell) \\
&= a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n
\end{aligned}
$$

Associativity (2)

$AB\ m \times \ell \quad C\ \ell \times t \to (AB)C$

$m \times t$

Theorem: Associativity

Given a $m \times n$ matrix A, a $n \times \ell$ matrix B and a $\ell \times t$ matrix C. Then

$$(AB)C = A(BC)$$

$BC\ n \times t$

$A\ m \times n$

$\to A(BC)$

$m \times t$

(proof)

$$\begin{aligned} LHS &= ((AB)\vec{c}_1\ \dots\ (AB)\vec{c}_t) \\ &= (A(B\vec{c}_1)\ \dots\ A(B\vec{c}_t)) \\ &= A(B\vec{c}_1\ \dots\ B\vec{c}_t) \\ &= A(BC) = RHS \end{aligned}$$

The last Theorem

ABC

Multiplication of matrices expressed by rows

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

In case A is a row vector

In case $A = \mathbf{a} = (a_1\ \dots\ a_n)$, we have for a $n \times \ell$ matrix $X = (\vec{x}_1\ \dots\ \vec{x}_\ell)$

$$\mathbf{a}X = (\mathbf{a}\vec{x}_1\ \dots\ \mathbf{a}\vec{x}_\ell)$$

$$= a_1 x_1 + \cdots + a_n x_n$$

Multiplication of matrices expressed by rows

Multiplication of matrices

For $m \times n$ matrix $A = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix}$ and $n \times \ell$ matrix we have

$X = (\vec{x}_1 \cdots \vec{x}_\ell)$

$$AX = \begin{pmatrix} \mathbf{a}_1 X \\ \vdots \\ \mathbf{a}_m X \end{pmatrix}$$

In fact

$$AX = \begin{pmatrix} \mathbf{a}_1\vec{x}_1 & \ldots & \mathbf{a}_1\vec{x}_\ell \\ \vdots & & \vdots \\ \mathbf{a}_i\vec{x}_1 & \ldots & \mathbf{a}_i\vec{x}_\ell \\ \vdots & & \vdots \\ \mathbf{a}_m\vec{x}_1 & \ldots & \mathbf{a}_m\vec{x}_\ell \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 X \\ \vdots \\ \mathbf{a}_i X \\ \vdots \\ \mathbf{a}_m X \end{pmatrix}$$

$= \mathbf{a}_i X$

Special Matrices

$O_{m,n}$ Zero Matrix

$m \times n$

$$O_{m,n} = (\vec{0} \ \ldots \ \vec{0}) = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{pmatrix}$$

m rows
n colums

is called the zero matrix. It satisfies the identity

$$O_{m,n}X = O_{m,\ell}, \quad YO_{m,n} = O_{k,n}$$

for a $n \times \ell$ matrix X and a $k \times m$ matrix Y.

$O_{m,n}X = O_{m,\ell}$ follows from

$(\vec{0} \cdots \vec{0}) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$=$

$$O_{m,n}\vec{x} = x_1\vec{0} + \cdots + x_n\vec{0} = \vec{0}$$

and $YO_{m,n} = O_{k,n}$ from

$$Y\vec{0} = 0\vec{y}_1 + \cdots + 0\vec{y}_n = \vec{0}$$

Standard Unit Vectors

Standard Unit Vectors

In $\mathbf{R}^3$, we have

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

called the standard unit vectors.

For $m \times 3$ matrix $X = (\vec{x}_1 \ \vec{x}_2 \ \vec{x}_3)$, we have

$$\begin{aligned} X\vec{e}_1 &= 1 \cdot \vec{x}_1 + 0 \cdot \vec{x}_2 + 0 \cdot \vec{x}_3 = \vec{x}_1 \\ X\vec{e}_2 &= 0 \cdot \vec{x}_1 + 1 \cdot \vec{x}_2 + 0 \cdot \vec{x}_3 = \vec{x}_2 \\ X\vec{e}_3 &= 0 \cdot \vec{x}_1 + 0 \cdot \vec{x}_2 + 1 \cdot \vec{x}_3 = \vec{x}_3 \end{aligned}$$

$$(\alpha \ \beta \ \gamma) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha x_1 + \beta x_2 + \gamma x_3.$$

Standard Unit Vectors

Standard Unit Vectors

In $(\mathbf{R}^3)^*$, we have

$$\mathbf{e}_1 = (1 \quad 0 \quad 0), \quad \mathbf{e}_2 = (0 \quad 1 \quad 0), \quad \mathbf{e}_3 = (0 \quad 0 \quad 1)$$

also called the standard unit vectors.

For $3 \times n$ matrix $X = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}$, we have

$$\begin{aligned} \mathbf{e}_1 X &= 1 \cdot \mathbf{x}_1 + 0 \cdot \mathbf{x}_2 + 0 \cdot \mathbf{x}_3 = \mathbf{x}_1 \\ \mathbf{e}_2 X &= 0 \cdot \mathbf{x}_1 + 1 \cdot \mathbf{x}_2 + 0 \cdot \mathbf{x}_3 = \mathbf{x}_2 \\ \mathbf{e}_3 X &= 0 \cdot \mathbf{x}_1 + 0 \cdot \mathbf{x}_2 + 1 \cdot \mathbf{x}_3 = \mathbf{x}_3 \end{aligned}$$

Special Matrices 2

I_n Identity Matrix

$n \times n$ matrix

$$I_n = (\vec{e}_1 \ \ldots \ \vec{e}_j \ \ldots \ \vec{e}_n) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & 1 \end{pmatrix}$$

I_n $n \times n$

is called the **Identity Matrix**. It enjoys, the identity

$$XI_n = X, \quad I_nY = Y$$

for a $m \times n$ matrix X and a $n \times \ell$ matrix Y.

$X\vec{e}_j = \vec{x}_j$

Special Matrices 2

We have the identity

$$(\vec{x}_1 \ \ldots \ \vec{x}_j \ \ldots \ \vec{x}_n)\vec{e}_j = 0 \cdot \vec{x}_1 + \cdots + 1 \cdot \vec{x}_j + \cdots + 0\vec{x}_n = \vec{x}_j$$

This leads us to

$$XI_n = (\vec{x}_1 \ \ldots \ \vec{x}_n)(\vec{e}_1 \ \ldots \vec{e}_n) = (\vec{x}_1 \ \ldots \ \vec{x}_n) = X$$

On the other hand we have, for $\vec{y} \in \mathbf{R}^n$, the identity

$$\vec{y} = y_1\vec{e}_1 + \cdots + y_n\vec{e}_n = I_n\vec{y}$$

Thus we get the identity

$$I_nY = (I_n\vec{y}_1 \ \ldots \ I_n\vec{y}_\ell) = (\vec{y}_1 \ \ldots \ \vec{y}_\ell) = Y$$

A : 2×2 matrix

A regular iff. there exists X : 2×2
$AX = XA = I_2$

Regularity of Matrices, Uniqueness of Inverse

Regularity of Matrices

A $n \times n$ matrix A is called **regular** if there exists another $n \times n$ matrix X satisfying
$$AX = XA = I_n$$
In this situation X is called the **inverse** of A.

- **(i) (Uniqueness of the inverse)** Assume
$$AX = XA = I_n, \quad AY = YA = I_n$$
Then $X = Y$. In fact, from $AX = I_n$ multiplied by Y from the left follows
$$Y(AX) = YI_n = Y$$
On the other hand, $Y(AX) = (YA)X = I_nX = X$. Accordingly $X = Y$.

How to characterize regularity?

How to calculate the inverse.

Elementary Matrices (1)

To exchange two rows and two columns

$$S_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\vec{e}_2\ \vec{e}_1\ \vec{e}_3) = \begin{pmatrix} \mathbf{e}_2 \\ \mathbf{e}_1 \\ \mathbf{e}_3 \end{pmatrix}$$

satisfies

$$S_{12}\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \\ \mathbf{c} \end{pmatrix}, \quad (\vec{a}\ \vec{b}\ \vec{c})S_{12} = (\vec{b}\ \vec{a}\ \vec{c})$$

$$\begin{pmatrix} \boxed{0\ 1\ 0} \\ \boxed{1\ 0\ 0} \\ \boxed{0\ 0\ 1} \end{pmatrix}\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \\ \mathbf{c} \end{pmatrix}$$

$0 \cdot \mathbf{a} + 1 \cdot \mathbf{b} + 0 \cdot \mathbf{c} = \mathbf{b}$

$1 \cdot \mathbf{a} + 0 \cdot \mathbf{b} + 0 \cdot \mathbf{c} = \mathbf{a}$

$0 \cdot \mathbf{a} + 0 \cdot \mathbf{b} + 1 \cdot \mathbf{c} = \mathbf{c}$

Elementary Matrices (1)

To exchange two rows and two columns

$$S_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (\vec{e}_3\ \vec{e}_2\ \vec{e}_1) = \begin{pmatrix} \mathbf{e}_3 \\ \mathbf{e}_2 \\ \mathbf{e}_1 \end{pmatrix}$$

satisfies

$$S_{13} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{c} \\ \mathbf{b} \\ \mathbf{a} \end{pmatrix}, \quad (\vec{a}\ \vec{b}\ \vec{c})S_{12} = (\vec{c}\ \vec{b}\ \vec{a})$$

What do we have for

$$S_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = (\vec{e}_1\ \vec{e}_3\ \vec{e}_2) = \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_3 \\ \mathbf{e}_2 \end{pmatrix} \quad ?$$

$$\begin{pmatrix} \boxed{0\ 0\ 1} \\ \boxed{0\ 1\ 0} \\ \boxed{1\ 0\ 0} \end{pmatrix} \begin{pmatrix} \mathbb{a} \\ \mathbb{b} \\ \mathbb{c} \end{pmatrix} = \begin{pmatrix} \mathbb{c} \\ \mathbb{b} \\ \mathbb{a} \end{pmatrix}$$

$$S_{23} \begin{pmatrix} \mathbb{a} \\ \mathbb{b} \\ \mathbb{c} \end{pmatrix} = ?$$

Elementary Matrices (1)

To exchange two rows and two columns

S_{12}, S_{13}, S_{23} are regular.

For example

$$S_{13}S_{13} = S_{13} \begin{pmatrix} \mathbf{e}_3 \\ \mathbf{e}_2 \\ \mathbf{e}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = I_3$$

S_{13} is regular

$$(S_{13})^{-1} = S_{13}.$$

Elementary Matrices (2)

$1 \cdot \vec{e_1} + \lambda \vec{e_2}$

To add $\lambda\times$ ith row to jth row

$$R_{21}(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\vec{e}_1 + \lambda\vec{e}_2\ \vec{e}_2\ \vec{e}_3) = \begin{pmatrix} \mathbf{e}_1 \\ \lambda\mathbf{e}_1 + \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

$\lambda \mathbf{e}_1 + \mathbf{e}_2$

satisfies

$$R_{21}(\lambda)\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \lambda\mathbf{a} + \mathbf{b} \\ \mathbf{c} \end{pmatrix}, \quad (\vec{a}\ \vec{b}\ \vec{c})R_{21}(\lambda) = (\vec{a} + \lambda\vec{b}\ \vec{b}\ \vec{c})$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \lambda\mathbf{a} + \mathbf{b} \\ \mathbf{c} \end{pmatrix}$$

to add $\lambda \times$ 1st row to 2nd row

$\mathbf{a}$

Elementary Matrices (2)

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} = \vec{e_1} + \lambda \vec{e_3}$

To add $\lambda\times$ ith row to jth row

$$R_{31}(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix} = (\vec{e}_1 + \lambda\vec{e}_3\ \vec{e}_2\ \vec{e}_3) = \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \lambda\mathbf{e}_1 + \mathbf{e}_3 \end{pmatrix}$$

$= \lambda(1\ 0\ 0) + (0\ 0\ 1)$

satisfies

$$R_{31}(\lambda)\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \lambda\mathbf{a} + \mathbf{c} \end{pmatrix}, \quad (\vec{a}\ \vec{b}\ \vec{c})R_{31}(\lambda) = (\vec{a} + \lambda\vec{c}\ \vec{b}\ \vec{c})$$

$\lambda \times$ the 1st row added to the 3rd.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \end{pmatrix}$$

$\lambda \cdot \mathbf{a} + 0 \cdot \mathbf{b} + 1 \cdot \mathbf{c}$

$= \lambda \mathbf{a} + \mathbf{c}$

$$\begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \mu & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \hline & & \\ 0 & 0 & 1 \end{pmatrix}$$

$\lambda(1\ 0\ 0)$
$+(\mu\ 1\ 0)$
$=(\lambda+\mu\ 1\ 0)$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \lambda+\mu & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Elementary Matrices (2)

To add $\lambda\times$ ith row to jth row

3×3 matrices $R_{ij}(\lambda)$ $(i \neq j)$ are all regular.

For example we have

$$R_{12}(\lambda)R_{12}(\mu) = R_{12}(\lambda+\mu), \quad R_{12}(0) = I_3$$

Thus we get

$$R_{12}(\lambda)R_{12}(-\lambda) = R_{12}(-\lambda)R_{12}(\lambda) = R_{12}(0) = I_3$$

$R_{12}(\lambda)$ is regular $(R_{12}(\lambda))^{-1} = R_{12}(-\lambda)$

Next week

Elementary Matrices (3)

To multiply jth row by $\lambda \neq 0$

$$Q_2(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\vec{e}_1\ \lambda\vec{e}_2\ \vec{e}_3) = \begin{pmatrix} \mathbf{e}_1 \\ \lambda\mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

satisfies

$$Q_2(\lambda)\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \lambda\mathbf{b} \\ \mathbf{c} \end{pmatrix}, \quad (\vec{a}\ \vec{b}\ \vec{c})Q_2(\lambda) = (\vec{a}\ \lambda\vec{b}\ \vec{c})$$