oct 25, 2016

I (1) $\Phi_A(\lambda) = \begin{vmatrix} \lambda-7 & -4 \\ 1 & \lambda-3 \end{vmatrix} = (\lambda-7)(\lambda-3)+4$

$$= \lambda^2 - 10\lambda + 25 = (\lambda-5)^2.$$

We divide λ^n by $(\lambda-5)^2$:

$$\lambda^n = g(\lambda)(\lambda-5)^2 + a\lambda + b \quad \cdots (1)$$

we derivate (1) by

$$n\lambda^{n-1} = g'(\lambda)(\lambda-5)^2 + g(\lambda)5(\lambda-5) + a \cdots (2)$$

We substitute 5 for λ in (1) and (2) to get

$$5^n = 5a + b \quad \cdots (1)'$$

$$n5^{n-1} = a \quad \cdots (2)'$$

Thus

$$a = n5^{n-1}, \quad b = 5^n - n5^n$$

and

$$\lambda^n = g(\lambda)(\lambda-5)^2 + n5^{n-1}\lambda + (5^n - n5^n).$$

we substitute A for λ to get

$$A^n = g(A)(A-5I_2)^2 + n5^{n-1}A + (5^n - n5^n)I_2$$

$$= n5^{n-1}A + 5^n(1-n)I_2.$$

Here we used C-H:

$$(A-5I_2)^2 = O_2.$$

(2) $$\Phi_A(\lambda) = \begin{vmatrix} \lambda-5 & 1 \\ -9 & \lambda+1 \end{vmatrix} = (\lambda-5)(\lambda+1)+9$$

$$= \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$$

We devide λ^n by $(\lambda-2)^2$:

$$\lambda^n = g(\lambda)\,(\lambda-2)^2 + a\lambda + b \quad \cdots (1)$$

We derivate (1) by

$$n\lambda^{n-1} = g'(\lambda)(\lambda-2)^2 + g(\lambda)\cdot 2(\lambda-2) + a$$
$$\cdots (2)$$

We substitute 2 for λ in (1) and (2) to get

$$2^n = 2a + b \quad \cdots (1)'$$
$$n\,2^{n-1} = a \quad \cdots (2)'$$

Thus

$$a = n\,2^{n-1}, \quad b = 2^n - n\,2^n = (1-n)\,2^n,$$

and

$$\lambda^n = g(\lambda)(\lambda-2)^2 + n\,2^{n-1}\lambda + (1-n)\,2^n.$$

We substitute A for λ to get

$$A^n = g(A)(A-2I_2)^2 + n\,2^{n-1}A + 2^n(1-n)I_2$$
$$= n\,2^{n-1}A + 2^n(1-n)I_2$$

Here we used C-H.

$$(A - 2I_2)^2 = O_2$$

II (1) $\Phi_A(\lambda) = \begin{vmatrix} \lambda-1 & -2 \\ -3 & \lambda+4 \end{vmatrix} = (\lambda-1)(\lambda+4) - 6$

$$= \lambda^2 + 3\lambda - 10 = (\lambda+5)(\lambda-2)$$

We divide λ^n by $(\lambda+5)(\lambda-2)$

$$\lambda^n = g(\lambda)(\lambda+5)(\lambda-2) + a\lambda + b \qquad (1)$$

We substitute -5 and 2 for λ to get

$$(-5)^n = -5a + b \qquad (2)$$

$$2^n = 2a + b \qquad (3)$$

Then (3) − (2) leads us

$$a = \frac{1}{7}(2^n - (-5)^n)$$

and

$$b = 2^n - \frac{2}{7}(2^n - (-5)^n)$$

$$= \frac{5}{7} 2^n + \frac{2}{7}(-5)^n.$$

Thus we find

$$\lambda^n = g(\lambda)(\lambda+5)(\lambda-2) + \frac{1}{7}(2^n - (-5)^n)\lambda + \frac{5}{7}2^n + \frac{2}{7}(-5)^n$$

We substitute A for λ to get

$$A^n = g(A)(A+5I_2)(A-2I_2) + \frac{1}{7}(2^n - (-5)^n)A + \left(\frac{5}{7}2^n + \frac{2}{7}(-5)^n\right)I_2$$

$$= \frac{1}{7}(2^n - (-5)^n)A + \left(\frac{5}{7}2^n + \frac{2}{7}(-5)^n\right)I_2$$

Here we used the identity derived by C-H,

$$(A+5I_2)(A-2I_2) = \Phi_A(A) = O_2$$

(2)

$$\Phi_A(\lambda) = \begin{vmatrix} \lambda+4 & 2 \\ -3 & \lambda-1 \end{vmatrix} = (\lambda+4)(\lambda-1)+6$$

$$= \lambda^2 + 3\lambda + 2$$

$$= (\lambda+1)(\lambda+2)$$

We devide λ^n by $(\lambda+1)(\lambda+2)$

$$\lambda^n = g(\lambda)(\lambda+1)(\lambda+2) + a\lambda + b, \quad \cdots (1)$$

We substitute -1 and -2 for λ to get

$$(-1)^n = -a + b \qquad \cdots (2)$$

$$(-2)^n = -2a + b \qquad \cdots (3)$$

Then (2) − (3) leads us to

$$a = (-1)^n - (-2)^n$$

and

$$b = (-1)^n + \{(-1)^n - (-2)^n\}$$

$$= 2\cdot(-1)^n + (-2)^n$$

Thus we find

$$\lambda^n = g(\lambda)(\lambda+1)(\lambda+2) + \left((-1)^n - (-2)^n\right)\lambda$$

$$+ 2(-1)^n + (-2)^n$$

We substitute A for λ to get

$$A^n = g(A)(A+I_2)(A+2I_2)$$

$$+ \left((-1)^n - (-2)^n\right)A + \left(2(-1)^n + (-2)^n\right)I_2$$

$$= \left((-1)^n - (-2)^n\right)A + \left(2(-1)^n + (-2)^n\right)I_2$$

Here we made use of the identity deduced by C-H,

$$(A+I_2)(A+2I_2) = \Phi_A(A) = O_2$$

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III

(1) $\Phi_A(\lambda) = \begin{vmatrix} \lambda-7 & -4 \\ 1 & \lambda-3 \end{vmatrix} = (\lambda-7)(\lambda-3)+4$

$$= \lambda^2 - 10\lambda + 25$$
$$= (\lambda-5)^2$$

Thus we have by C-H

$$A^2 - (5+5)A + 5^2 I_2 = O_2.$$

which is

$$A(A-5I_2) = 5(A-5I_2) \quad \cdots (1)$$

By applying the identity (1) repeatedly we get

$$A^n(A-5I_2) = 5^n(A-5I_2).$$

which we view as

$$A^{n+1} - 5A^n = 5^n(A-5I_2)$$

By dividing this by 5^{n+1} it follows

$$\frac{1}{5^{n+1}}A^{n+1} - \frac{1}{5^n}A^n = \frac{1}{5}(A-5I_2) \quad \cdots (2)$$

Then it follows from (2) that

$$\frac{1}{5^n}A^n = I_2 + \frac{n}{5}(A-5I_2)$$

Accordingly we have shown

$$A^n = 5^n I_2 + n5^{n-1}(A-5I_2)$$

(2)

$$\Phi_A(\lambda) = \begin{vmatrix} \lambda-5 & 1 \\ -9 & \lambda+1 \end{vmatrix} = (\lambda-5)(\lambda+1)+9$$

$$= \lambda^2 - 4\lambda + 4 = (\lambda-2)^2.$$

Thus we have by C-H

$$A^2 - (2+2)A + 2^2 I_2 = O_2$$

which is

$$A(A - 2I_2) = 2(A - 2I_2). \quad \cdots (1)$$

By applying the identity (1) repeatedly we get

$$A^n(A - 2I_2) = 2^n(A - 2I_2).$$

which we view as

$$A^{n+1} - 2A^n = 2^n(A - 2I_2).$$

By dividing this by 2^{n+1} it follows that

$$\frac{1}{2^{n+1}} A^{n+1} - \frac{1}{2^n} A^n = \frac{1}{2}(A - 2I_2) \quad \cdots (2)$$

Then it follows from (2) that

$$\frac{1}{2^n} A^n = I_2 + \frac{n}{2}(A - 2I_2).$$

Accordingly we have shown

$$A^n = 2^n I_2 + n2^{n-1}(A - 2I_2)$$

II (3) $\Phi_A(\lambda) = \begin{vmatrix} \lambda-1 & -12 \\ -3 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 - 36$

$$= (\lambda+5)(\lambda-7)$$

We divide λ^n by $(\lambda+5)(\lambda-7)$

$$\lambda^n = g(\lambda)(\lambda+5)(\lambda-7) + a\lambda + b. \qquad (1)$$

We substitute -5 and 7 for λ to get

$$(-5)^n = -5a + b \qquad (2)$$
$$7^n = 7a + b \qquad (3)$$

Then $(3) - (2)$ leads us to

$$12a = 7^n - (-5)^n \quad \text{i.e.} \quad a = \frac{1}{12}(7^n - (-5)^n)$$

and

$$\begin{aligned} b &= 5a + (-5)^n \\ &= \frac{5}{12}(7^n - (-5)^n) + (-5)^n \\ &= \frac{5}{12}7^n + \frac{7}{12}(-5)^n. \end{aligned}$$

Thus we have shown

$$\lambda^n = g(\lambda)(\lambda+5)(\lambda-7) + \frac{1}{12}(7^n - (-5)^n)\lambda + \frac{5}{12}7^n + \frac{7}{12}(-5)^n$$

We substitute A for λ to get

$$\begin{aligned} A^n &= g(A)(A+5I_2)(A-5I_2) + \frac{1}{12}(7^n-(-5)^n)A + \left(\frac{5}{12}7^n + \frac{7}{12}(-5)^n\right)I_2 \\ &= \frac{1}{12}(7^n-(-5)^n)A + \left(\frac{5}{12}7^n + \frac{7}{12}(-5)^n\right)I_2 \end{aligned}$$

Here we used the identity derived by C-H

$$(A+5I_2)(A-7I_2) = \Phi_A(A) = O_2.$$

IV $\Phi_A(\lambda) = \begin{vmatrix} \lambda-1 & -1 \\ -3 & \lambda-2 \end{vmatrix} = (\lambda-1)(\lambda-2) - 3$

$$= \lambda^2 - 3\lambda + 2$$

We devide $f(\lambda) := \lambda^3 - \lambda^2 - 8\lambda + 1$ by $\Phi_A(\lambda)$ to get

$$f(\lambda) = (\lambda^2 - 3\lambda + 2)\,\Phi_A(\lambda) - 3.$$

$$\begin{array}{r|rrrr}
 & 1 & 2 & & \\ \hline
1\ -3\ 2 & 1 & -1 & -8 & 1 \\
 & 1 & -3 & 2 & \\ \hline
 & & 2 & -6 & 1 \\
 & & 2 & -6 & 4 \\ \hline
 & & & & -3.
\end{array}$$

We substitute A for λ and get

$$f(A) = (A^2 - 3A + 2I_2)\,\Phi_A(A) - 3I_2$$

$$= -3I_2.$$

V (i) $\begin{pmatrix} x+\lambda z & p+\lambda r \\ y & q \\ z & r \end{pmatrix}$ (ii) $\begin{pmatrix} x & p \\ \lambda y + z & \lambda p + q \\ z & r \end{pmatrix}$

(iii) $\begin{pmatrix} x & p \\ y+\lambda z & q+\lambda r \\ z & r \end{pmatrix}$ (iv) $\begin{pmatrix} z & r \\ y & q \\ x & p \end{pmatrix}$

(v) $\begin{pmatrix} y & q \\ x & p \\ z & r \end{pmatrix}$ (vi) $\begin{pmatrix} x & p \\ z & r \\ y & q \end{pmatrix}$

(vii) $\begin{pmatrix} x & p \\ \lambda y & \lambda q \\ z & r \end{pmatrix}$ (viii) $\begin{pmatrix} x & p \\ y & q \\ \lambda z & \lambda r \end{pmatrix}$

(ix) $\begin{pmatrix} x+\lambda\alpha & p & \alpha \\ y+\lambda\beta & q & \beta \\ z+\lambda\gamma & r & \gamma \end{pmatrix}$ (x) $\begin{pmatrix} x+\lambda p & p & \alpha \\ y+\lambda q & q & \beta \\ z+\lambda r & r & \gamma \end{pmatrix}$

(xi) $\begin{pmatrix} x & p & \lambda p+\alpha \\ y & q & \lambda q+\beta \\ z & r & \lambda r+\gamma \end{pmatrix}$ (xii) $\begin{pmatrix} \lambda x & p & \alpha \\ \lambda y & q & \beta \\ \lambda z & r & \gamma \end{pmatrix}$

(xiii) $\begin{pmatrix} 1 & 0 & \lambda+\mu \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (xiv) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(xv) $\begin{pmatrix} ax & au+yd & av+dw+ez \\ 0 & by & bw+fz \\ 0 & 0 & cz \end{pmatrix}$

(xvi) $\begin{pmatrix} ax & 0 & 0 \\ dx+bu & by & 0 \\ ex+fu+cv & fy+cw & cz \end{pmatrix}$