Matrices and their operations No. 2

$m \times n$ Matrices

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$m \times n$ Matrices

$m \times n$ matrices: How to make them

$$A = (\vec{a}_1 \ \ \dots \ \ \vec{a}_n) = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

A $m \times n$ matrix is given in the following ways.

- (i) Combining n column vectors $\vec{a}_1, \ \dots, \ \vec{a}_n \in \mathbf{R}^m$
- (ii) Combining m row vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$
- (iii) Giving $m \times n$ components.

NB a_{ij} is used for the component of the ith row and of the jth column, and sometimes called (i, j) component.

Multiplication: Matrix × Column Vector

Multiplication of n-dim. col. vectors to $m \times n$ matrices

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} = x_1 \vec{a}_1 + \cdots + x_j \vec{a}_j + \cdots + x_n \vec{a}_n = \begin{pmatrix} \mathbf{a}_1 \vec{x} \\ \vdots \\ \mathbf{a}_i \vec{x} \\ \vdots \\ \mathbf{a}_m \vec{x} \end{pmatrix} \in \mathbf{R}^m$$

Here we use multiplication of row vectors and column vectros:

$$\begin{pmatrix} \alpha_1 & \ldots & \alpha_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \alpha_1 x_1 + \cdots + \alpha_n x_n$$

Multiplication: Matrix × Column Vector

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{a}_1 + \cdots + x_j\vec{a}_j + \cdots + x_n\vec{a}_n$$

$$= x_1\begin{pmatrix} a_{11} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + x_j\begin{pmatrix} a_{1j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix} + \cdots + x_n\begin{pmatrix} a_{1n} \\ \vdots \\ a_{in} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} \vdots \\ x_1 a_{i1} + \cdots + x_j a_{ij} + \cdots + x_n a_{in} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \mathbf{a}_i\vec{x} \\ \vdots \end{pmatrix}$$

Multiplication: Matrix × Matrix

Multiplication: Matrix × Matrix

Take another matrix of type $n \times \ell$:

$$X = (\vec{x}_1 \ \ldots \ \vec{x}_\ell) = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

Then a $m \times \ell$ matrix AX is defined by

$$AX = (A\vec{x}_1 \ \ldots \ A\vec{x}_\ell) = \begin{pmatrix} \mathbf{a}_1\vec{x}_1 & \ldots & \mathbf{a}_1\vec{x}_\ell \\ \vdots & & \vdots \\ \mathbf{a}_m\vec{x}_1 & \ldots & \mathbf{a}_m\vec{x}_\ell \end{pmatrix}$$

Linear Map defined by A

Linear Map defined by A

We can define a map

$$\begin{aligned} F_A : \mathbf{R}^n &\longrightarrow \mathbf{R}^m \\ \vec{x} &\mapsto A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n \end{aligned}$$

Linearity of F_A

Linearity of F_A

F_A satisfies the following basic properties called **Linearity**.

- **(i)** $F_A(\vec{x}+\vec{y}) = F_A(\vec{x}) + F_A(\vec{y})$
- **(ii)** $F_A(\lambda\vec{x}) = \lambda F_A(\vec{x})$
- **(iii)** $F_A(\lambda\vec{x}+\mu\vec{y}) = \lambda F_A(\vec{x}) + \mu F_A(\vec{y})$

These three propeties are identical to the following.

- **(i)** $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$
- **(ii)** $A(\lambda\vec{x}) = \lambda(A\vec{x})$
- **(iii)** $A(\lambda\vec{x}+\mu\vec{y}) = \lambda(A\vec{x}) + \mu(A\vec{y})$

Moreover remark that **(iii)** can be easily derived from **(i)** and **(ii)**.

$$A(\lambda\vec{x}+\mu\vec{y}) = A(\lambda\vec{x}) + A(\mu\vec{y}) = \lambda(A\vec{x}) + \mu(A\vec{y})$$

Proof

Proof for (i)

$$LHS = A\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\right) = A\begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

$$\begin{aligned}
&= (x_1 + y_1)\vec{a}_1 + \cdots + (x_n + y_n)\vec{a}_n \\
&= x_1\vec{a}_1 + y_1\vec{a}_1 + \cdots + x_n\vec{a}_n + y_n\vec{a}_n \\
&= (x_1\vec{a}_1 + \cdots + x_n\vec{a}_n) + (y_1\vec{a}_1 + \cdots + y_n\vec{a}_n) \\
&= A\vec{x} + A\vec{y} = RHS
\end{aligned}$$

Proof for (ii)

$$LHS = A\begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix} = (\lambda x_1)\vec{a}_1 + \cdots + (\lambda x_n)\vec{a}_n$$

$$= \lambda(x_1\vec{a}_1) + \cdots + \lambda(x_n\vec{a}_n) = \lambda(x_1\vec{a}_1 + \cdots + x_n\vec{a}_n) = RHS$$

Corollary

Corollary

Let A be a $m \times n$ matrix, $\vec{x}_1, \ldots, \vec{x}_\ell \in \mathbf{R}^n$. Then we have

- **(i)** $A(\vec{x}_1 + \cdots + \vec{x}_\ell) = A\vec{x}_1 + \cdots + A\vec{x}_\ell$
- **(ii)** $A(c_1\vec{x}_1 + \cdots + c_\ell\vec{x}_\ell) = c_1(A\vec{x}_1) + \cdots + c_\ell(A\vec{x}_\ell)$

(i) follows from linearity with the aid of induction on ℓ.

Associativity (1)

Thanks to the Linearity, we can prove the following theorem about **Associativity**.

Theorem: Associativity

Given a $m \times n$ matrix A and a $n \times \ell$ matrix X. Then we have for $\vec{c} \in \mathbf{R}^\ell$.

$$(AX)\vec{c} = A(X\vec{c})$$

In fact,

$$\begin{aligned} RHS &= A(c_1\vec{x}_1 + \cdots + c_n\vec{x}_n) \\ &= c_1(A\vec{x}_1) + \cdots + c_n(A\vec{x}_n) \\ &= (A\vec{x}_1 \ \ldots \ A\vec{x}_n)\vec{c} = (AX)\vec{c} = LHS \end{aligned}$$

Associativity (2)

Theorem: Associativity

Given a $m \times n$ matrix A, a $n \times \ell$ matrix B and a $\ell \times t$ matrix C. Then

$$(AB)C = A(BC)$$

(proof)

$$\begin{aligned} LHS &= ((AB)\vec{c}_1 \ \ldots \ (AB)\vec{c}_t) \\ &= (A(B\vec{c}_1) \ \ldots \ A(B\vec{c}_t)) \\ &= A(B\vec{c}_1 \ \ldots \ B\vec{c}_t) \\ &= A(BC) \end{aligned}$$

Multiplication of matrices expressed by rows

In case A is a row vector

In case $A = \mathbf{a} = (a_1 \ \ldots \ a_n)$, we have for a $n \times \ell$ matrix $X = (\vec{x}_1 \ \ldots \ \vec{x}_\ell)$

$$\mathbf{a}X = (\mathbf{a}\vec{x}_1 \ \ldots \ \mathbf{a}\vec{x}_\ell)$$

Multiplication of matrices expressed by rows

On the other hand

$$
\begin{aligned}
& a_1\mathbf{x}_1 + \cdots + a_i\mathbf{x}_i + \cdots + a_n\mathbf{x}_n \\
=& a_1(x_{11} \ \ldots \ x_{1j} \ \ldots \ x_{1\ell}) \\
& \qquad \vdots \\
& + a_i(x_{i1} \ \ldots \ x_{ij} \ \ldots \ x_{i\ell}) \\
& \qquad \vdots \\
& + a_n(x_{n1} \ \ldots \ x_{nj} \ \ldots \ x_{n\ell}) \\
=& (\ldots \ a_1x_{1j} + \cdots + a_ix_{ij} + \cdots + a_nx_{n\ell} \ \ldots) \\
=& (\mathbf{a}\vec{x}_1 \ \ldots \ \mathbf{a}\vec{x}_j \ \ldots \ \mathbf{a}\vec{x}_\ell) \\
=& \mathbf{a}X
\end{aligned}
$$

Multiplication of matrices expressed by rows

In case A is a row vector

For a n dim. row vector $\mathbf{a} = (a_1 \ \dots \ a_n)$ and a $n \times \ell$ matrix

$$X = (\vec{x}_1 \ \dots \ \vec{x}_\ell) = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

we have

$$\begin{aligned} \mathbf{a}X &= (\mathbf{a}\vec{x}_1 \ \dots \ \mathbf{a}\vec{x}_\ell) \\ &= a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n \end{aligned}$$

Multiplication of matrices expressed by rows

Multiplication of matrices

For $m \times n$ matrix $A = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix}$ and $n \times \ell$ matrix we have

$$AX = \begin{pmatrix} \mathbf{a}_1 X \\ \vdots \\ \mathbf{a}_m X \end{pmatrix}$$

In fact

$$AX = \begin{pmatrix} \mathbf{a}_1\vec{x}_1 & \dots & \mathbf{a}_1\vec{x}_\ell \\ \vdots & & \vdots \\ \mathbf{a}_i\vec{x}_1 & \dots & \mathbf{a}_i\vec{x}_\ell \\ \vdots & & \vdots \\ \mathbf{a}_m\vec{x}_1 & \dots & \mathbf{a}_m\vec{x}_\ell \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 X \\ \vdots \\ \mathbf{a}_i X \\ \vdots \\ \mathbf{a}_m X \end{pmatrix}$$

Special Matrices

$O_{m,n}$ Zero Matrix

$$O_{m,n} = (\vec{0} \ \dots \ \vec{0}) = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

is called the zero matrix. It satisfies the identity

$$O_{m,n}X = O_{m,\ell}, \quad YO_{m,n} = O_{k,n}$$

for a $n \times \ell$ matrix X and a $k \times m$ matrix Y.

$O_{m,n}X = O_{m,\ell}$ follows from

$$O_{m,n}\vec{x} = x_1\vec{0} + \cdots + x_n\vec{0} = \vec{0}$$

and $YO_{m,n} = O_{k,n}$ from

$$Y\vec{0} = 0\vec{y}_1 + \cdots + 0\vec{y}_n = \vec{0}$$

Standard Unit Vectors

Standard Unit Vectors

In $\mathbf{R}^3$, we have

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

called the standard unit vectors.

For $m \times 3$ matrix $X = (\vec{x}_1 \ \vec{x}_2 \ \vec{x}_3)$, we have

$$\begin{aligned} X\vec{e}_1 &= 1 \cdot \vec{x}_1 + 0 \cdot \vec{x}_2 + 0 \cdot \vec{x}_3 = \vec{x}_1 \\ X\vec{e}_2 &= 0 \cdot \vec{x}_1 + 1 \cdot \vec{x}_2 + 0 \cdot \vec{x}_3 = \vec{x}_2 \\ X\vec{e}_3 &= 0 \cdot \vec{x}_1 + 0 \cdot \vec{x}_2 + 1 \cdot \vec{x}_3 = \vec{x}_3 \end{aligned}$$

Standard Unit Vectors

Standard Unit Vectors

In $\left(\mathbf{R}^3\right)^*$, we have

$$\mathbf{e}_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

also called the standard unit vectors.

For $3 \times n$ matrix $X = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}$, we have

$$\mathbf{e}_1 X = 1 \cdot \mathbf{x}_1 + 0 \cdot \mathbf{x}_2 + 0 \cdot \mathbf{x}_3 = \mathbf{x}_1$$
$$\mathbf{e}_2 X = 0 \cdot \mathbf{x}_1 + 1 \cdot \mathbf{x}_2 + 0 \cdot \mathbf{x}_3 = \mathbf{x}_2$$
$$\mathbf{e}_3 X = 0 \cdot \mathbf{x}_1 + 0 \cdot \mathbf{x}_2 + 1 \cdot \mathbf{x}_3 = \mathbf{x}_3$$

Special Matrices 2

I_n Identity Matrix

$$I_n = (\vec{e}_1 \ \ldots \ \vec{e}_j \ \ldots \ \vec{e}_n) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & 1 \end{pmatrix}$$

is called the **Identity Matrix**. It enjoys, the identity

$$XI_n = X, \quad I_n Y = Y$$

for a $m \times n$ matrix X and a $n \times \ell$ matrix Y.

Special Matrices 2

We have the identity

$$(\vec{x}_1 \ \dots \ \vec{x}_j \ \dots \ \vec{x}_n)\vec{e}_j = 0 \cdot \vec{x}_1 + \cdots + 1 \cdot \vec{x}_j + \cdots + 0\vec{x}_n = \vec{x}_j$$

This leads us to

$$XI_n = (\vec{x}_1 \ \dots \ \vec{x}_n)(\vec{e}_1 \ \dots \vec{e}_n) = (\vec{x}_1 \ \dots \ \vec{x}_n) = X$$

On the other hand we have, for $\vec{y} \in \mathbf{R}^n$, the identity

$$\vec{y} = y_1\vec{e}_1 + \cdots + y_n\vec{e}_n = I_n\vec{y}$$

Thus we get the identity

$$I_nY = (I_n\vec{y}_1 \ \dots \ I_n\vec{y}_\ell) = (\vec{y}_1 \ \dots \ \vec{y}_\ell) = Y$$

Regularity of Matrices, Uniqueness of Inverse

Regularity of Matrices

A $n \times n$ matrix A is called **regular** if there exists another $n \times n$ matrix X satisfying

$$AX = XA = I_n$$

In this situation X is called the **inverse** of A.

- **(i) (Uniqueness of the inverse)** Assume

  $$AX = XA = I_n, \quad AY = YA = I_n$$

  Then $X = Y$. In fact, from $AX = I_n$ multiplied by Y from the left follows

  $$Y(AX) = YI_n = Y$$

  On the other hand, $Y(AX) = (YA)X = I_nX = X$.
  Accordingly $X = Y$.

Elementary Matrices (1)

To exchange two rows and two columns

$$S_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\vec{e}_2\ \vec{e}_1\ \vec{e}_3) = \begin{pmatrix} \mathbf{e}_2 \\ \mathbf{e}_1 \\ \mathbf{e}_3 \end{pmatrix}$$

satisfies

$$S_{12} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \\ \mathbf{c} \end{pmatrix}, \quad (\vec{a}\ \vec{b}\ \vec{c})S_{12} = (\vec{b}\ \vec{a}\ \vec{c})$$

Elementary Matrices (1)

To exchange two rows and two columns

$$S_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (\vec{e}_3 \ \vec{e}_2 \ \vec{e}_1) = \begin{pmatrix} \mathbf{e}_3 \\ \mathbf{e}_2 \\ \mathbf{e}_1 \end{pmatrix}$$

satisfies

$$S_{13} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{c} \\ \mathbf{b} \\ \mathbf{a} \end{pmatrix}, \quad (\vec{a} \ \vec{b} \ \vec{c})S_{12} = (\vec{c} \ \vec{b} \ \vec{a})$$

What do we have for

$$S_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = (\vec{e}_1 \ \vec{e}_3 \ \vec{e}_2) = \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_3 \\ \mathbf{e}_2 \end{pmatrix} \quad ?$$

Elementary Matrices (1)

To exchange two rows and two columns

S_{12}, S_{13}, S_{23} are regular.

For example

$$S_{13}S_{13} = S_{13}\begin{pmatrix} \mathbf{e}_3 \\ \mathbf{e}_2 \\ \mathbf{e}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = I_3$$

Elementary Matrices (2)

To add $\lambda\times$ ith row to jth row

$$R_{21}(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\vec{e}_1 + \lambda\vec{e}_2 \ \vec{e}_2 \ \vec{e}_3) = \begin{pmatrix} \mathbf{e}_1 \\ \lambda\mathbf{e}_1 + \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

satisfies

$$R_{21}(\lambda)\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \lambda\mathbf{a} + \mathbf{b} \\ \mathbf{c} \end{pmatrix}, \quad (\vec{a} \ \vec{b} \ \vec{c})R_{21}(\lambda) = (\vec{a} + \lambda\vec{b} \ \vec{b} \ \vec{c})$$

Elementary Matrices (2)

To add $\lambda \times$ ith row to jth row

$$R_{31}(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix} = (\vec{e}_1 + \lambda\vec{e}_3 \ \vec{e}_2 \ \vec{e}_3) = \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \lambda\mathbf{e}_1 + \mathbf{e}_3 \end{pmatrix}$$

satisfies

$$R_{31}(\lambda)\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \lambda\mathbf{a} + \mathbf{c} \end{pmatrix}, \quad (\vec{a} \ \vec{b} \ \vec{c})R_{31}(\lambda) = (\vec{a} + \lambda\vec{c} \ \vec{b} \ \vec{c})$$

Elementary Matrices (2)

To add $\lambda\times$ ith row to jth row

3×3 matrices $R_{ij}(\lambda)$ $(i \neq j)$ are all regular.

For example we have

$$R_{12}(\lambda)R_{12}(\mu) = R_{12}(\lambda+\mu), \quad R_{12}(0) = I_3$$

Thus we get

$$R_{12}(\lambda)R_{12}(-\lambda) = R_{12}(-\lambda)R_{12}(\lambda) = R_{12}(0) = I_3$$

Elementary Matrices (3)

To multiply jth row by $\lambda \neq 0$

$$Q_2(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\vec{e}_1 \ \lambda\vec{e}_2 \ \vec{e}_3) = \begin{pmatrix} \mathbf{e}_1 \\ \lambda\mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

satisfies

$$Q_2(\lambda) \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \lambda\mathbf{b} \\ \mathbf{c} \end{pmatrix}, \quad (\vec{a} \ \vec{b} \ \vec{c})Q_2(\lambda) = (\vec{a} \ \lambda\vec{b} \ \vec{c})$$

Elementary Matrices (3)

To multiply jth row by $\lambda \neq 0$

$Q_i(\lambda)$ $(i = 1, 2, 3)$ are regular.

In fact we have

$$Q_2(\lambda)Q_2(\mu) = Q_2(\lambda\mu), \quad Q_2(1) = I_3$$

Thus

$$Q_2(\lambda)Q_2(\frac{1}{\lambda}) = Q_2(\frac{1}{\lambda})Q_2(\lambda) = Q_2(1) = I_3$$

Scalar Multiplication to Matirces

Scalar Multiplication to $m \times n$ Matrices

Given a $m \times n$ matix

$$A = (\vec{a}_1 \ \ \ldots \ \ \vec{a}_n) = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix},$$

we define a scalar multiplication by λ to A as follows:

$$\lambda A = (\lambda\vec{a}_1 \ \ \ldots \ \ \lambda\vec{a}_n) = \begin{pmatrix} \lambda\mathbf{a}_1 \\ \vdots \\ \lambda\mathbf{a}_m \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \vdots & & \vdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{pmatrix}$$

Scalar Multiplication to Matirces

Theorem

- **(i)** $(\lambda A)\vec{x} = \lambda(A\vec{x}) = A(\lambda\vec{x})$
- **(ii)** $(\lambda A)X = \lambda(AX) = A(\lambda X)$

The proof for **(i)** is given as follows:

$$
\begin{aligned}
(\lambda A)\vec{x} &= (\lambda\vec{a}_1 \ \ldots \ \lambda\vec{a}_n)\vec{x} \\
&= x_1(\lambda\vec{a}_1) + \cdots + x_n(\lambda\vec{a}_n) = (*) \\
&= \lambda(x_1\vec{a}_1) + \cdots + \lambda(x_n\vec{a}_n) \\
&= \lambda(x_1\vec{a}_1 + \cdots + x_n\vec{a}_n) = \lambda(A\vec{x}) \\
(*) &= (\lambda x_1)\vec{a}_1 + \cdots + (\lambda x_n)\vec{a}_n = A(\lambda\vec{x})
\end{aligned}
$$

Moreover the property **(ii)** is derived easily from **(i)**.

Other Basic Properties of Scalar Multiplication

Theorem

- **(iii)** $(\lambda + \mu)A = \lambda A + \mu A$
- **(iv)** $(\lambda\mu)A = \lambda(\mu A)$
- **(v)** $1A = A$ and $0A = O_{m,n}$

These properties can derived from the following corresponding properties for vectors. It is necessary to define the addition of matrices to understand (iii), and we put it off for the moment.

- **(iii)** $(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$
- **(iv)** $(\lambda\mu)\vec{a} = \lambda(\mu\vec{a})$
- **(v)** $1\vec{a} = \vec{a}$ and $0\vec{a} = \vec{0}$

Addition of two $m \times n$ Matrices

Definition

Given two $m \times n$ matrices

$$A = (\vec{a}_1 \ \dots \ \vec{a}_n) = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{im} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

$$B = (\vec{b}_1 \ \dots \ \vec{b}_n) = \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_m \end{pmatrix} = \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ \vdots & & \vdots & & \vdots \\ b_{i1} & \cdots & b_{ij} & \cdots & b_{im} \\ \vdots & & \vdots & & \vdots \\ b_{m1} & \cdots & b_{mj} & \cdots & b_{mn} \end{pmatrix}$$

Addition of two $m \times n$ Matrices

Definition

Then

$$A + B = (\vec{a}_1 + \vec{b}_1 \ \ \dots \ \ \vec{a}_n + \vec{b}_n) = \begin{pmatrix} \mathbf{a}_1 + \mathbf{b}_1 \\ \vdots \\ \mathbf{a}_m + \mathbf{b}_m \end{pmatrix}$$

$$= \begin{pmatrix} & \vdots & \\ \cdots & a_{ij} + b_{ij} & \cdots \\ & \vdots & \end{pmatrix}$$

Basic Properties (1)

Basic Properties (1)

- **(i)** $(A+B)+C = A+(B+C)$
- **(ii)** $A + O_{m,n} = O_{m,n} + A = A$
- **(iii)** $A+B = B+A$
- **(iv)** $\lambda(A+B) = \lambda A + \lambda B$
- **(v)** $(\lambda+\mu)A = \lambda A + \mu A$

(i) follows from $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$.
(ii) follows from $\vec{a}+\vec{0} = \vec{0}+\vec{a} = \vec{a}$.
(iii) follows from $\vec{a}+\vec{b} = \vec{b}+\vec{a}$.
(v) follows from $(\lambda+\mu)\vec{a} = \lambda\vec{a}+\mu\vec{a}$. (iv) is proved as follows.

$$
\begin{aligned}
LHS &= \lambda(\ldots \vec{a}_j + \vec{b}_j \ldots) = (\ldots \lambda(\vec{a}_j + \vec{b}_j) \ldots) \\
&= (\ldots \lambda\vec{a}_j + \lambda\vec{b}_j \ldots) = (\ldots \lambda\vec{a}_j \ldots) + (\ldots \lambda\vec{b}_j \ldots) = RHS
\end{aligned}
$$

Basic Properties (2)

Basic Properties (2)

- **(vi)** For $n \times \ell$ matrices X and Y, we have
$$A(X+Y) = AX + AY$$
- **(vii)** For $s \times m$ matrices P and Q, we have
$$(P+Q)A = PA + QA$$

(vi) follows from $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$. In fact

$$\begin{aligned} LHS &= A(\ldots \vec{x}_k + \vec{y}_k \ldots) = (\ldots A(\vec{x}_k + \vec{y}_k) \ldots) \\ &= (\ldots A\vec{x}_k + A\vec{y}_k \ldots) = (\ldots A\vec{x}_k \ldots) + (\ldots A\vec{y}_k \ldots) = RHS \end{aligned}$$

Basic Properties (2)

(vii) follows from the identity $(P+Q)\vec{a} = P\vec{a} + Q\vec{a}$ which is derived in the following way.

$$
\begin{aligned}
LHS &= (\vec{p}_1 + \vec{q}_1 \cdots \vec{p}_m + \vec{q}_m)\vec{a} \\
&= a_1(\vec{p}_1 + \vec{q}_1) + \cdots + a_m(\vec{p}_m + \vec{q}_m) \\
&= \cdots = (a_1\vec{p}_1 + \cdots + a_m\vec{p}_m) + (a_1\vec{q}_1 + \cdots + a_m\vec{q}_m) \\
&= RHS
\end{aligned}
$$