Matrices and their operations No. 2

$m \times n$ Matrices

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$m \times n$ Matrices

$m \times n$ matrices: How to make them

$$A = (\vec{a}_1 \ \ \ldots \ \ \vec{a}_n) = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} = \begin{pmatrix} a_{11} & \ldots & a_{1j} & \ldots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \ldots & a_{ij} & \ldots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \ldots & a_{mj} & \ldots & a_{mn} \end{pmatrix}$$

A $m \times n$ matrix is given in the following ways.

- (i) Combining n column vectors $\vec{a}_1, \ \ldots, \ \vec{a}_n \in \mathbf{R}^m$
- (ii) Combining m row vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$
- (iii) Giving $m \times n$ components.

NB a_{ij} is used for the component of the ith row and of the jth column, and sometimes called (i, j) component.

Multiplication: Matrix × Column Vector

Multiplication of n-dim. col. vectors to $m \times n$ matrices

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{a}_1 + \cdots + x_j\vec{a}_j + \cdots + x_n\vec{a}_n = \begin{pmatrix} \mathbf{a}_1\vec{x} \\ \vdots \\ \mathbf{a}_i\vec{x} \\ \vdots \\ \mathbf{a}_m\vec{x} \end{pmatrix} \in \mathbf{R}^m$$

Here we use multiplication of row vectors and column vectros:

$$(\alpha_1 \ \ldots \ \alpha_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \alpha_1 x_1 + \cdots + \alpha_n x_n$$

Multiplication: Matrix $\times$ Column Vector

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{a}_1 + \cdots + x_j\vec{a}_j + \cdots + x_n\vec{a}_n$$

$$= x_1\begin{pmatrix} a_{11} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + x_j\begin{pmatrix} a_{1j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix} + \cdots + x_n\begin{pmatrix} a_{1n} \\ \vdots \\ a_{in} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} \vdots \\ x_1a_{i1} + \cdots + x_ja_{ij} + \cdots + x_na_{in} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \mathbf{a}_i\vec{x} \\ \vdots \end{pmatrix}$$

Multiplication: Matrix × Matrix

Multiplication: Matrix × Matrix

Take another matrix of type $n \times \ell$:

$$X = (\vec{x}_1 \;\; \dots \;\; \vec{x}_\ell) = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

Then a $m \times \ell$ matrix AX is defined by

$$AX = (A\vec{x}_1 \;\; \dots \;\; A\vec{x}_\ell) = \begin{pmatrix} \mathbf{a}_1\vec{x}_1 & \dots & \mathbf{a}_1\vec{x}_\ell \\ \vdots & & \vdots \\ \mathbf{a}_m\vec{x}_1 & \dots & \mathbf{a}_m\vec{x}_\ell \end{pmatrix}$$

Linear Map defined by A

We can define a map

$$\begin{aligned} F_A : \mathbf{R}^n &\longrightarrow \mathbf{R}^m \\ \vec{x} &\mapsto A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n \end{aligned}$$

Linearity of F_A

Linearity of F_A

F_A satisfies the following basic properties called **Linearity**.

- **(i)** $F_A(\vec{x}+\vec{y}) = F_A(\vec{x}) + F_A(\vec{y})$
- **(ii)** $F_A(\lambda\vec{x}) = \lambda F_A(\vec{x})$
- **(iii)** $F_A(\lambda\vec{x}+\mu\vec{y}) = \lambda F_A(\vec{x}) + \mu F_A(\vec{y})$

These three propeties are identical to the following.

- **(i)** $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$
- **(ii)** $A(\lambda\vec{x}) = \lambda(A\vec{x})$
- **(iii)** $A(\lambda\vec{x}+\mu\vec{y}) = \lambda(A\vec{x}) + \mu(A\vec{y})$

Moreover remark that **(iii)** can be easily derived from **(i)** and **(ii)**.

$$A(\lambda\vec{x}+\mu\vec{y}) = A(\lambda\vec{x}) + A(\mu\vec{y}) = \lambda(A\vec{x}) + \mu(A\vec{y})$$

Proof

Proof for (i)

$$LHS = A\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\right) = A\begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

$$= (x_1 + y_1)\vec{a}_1 + \cdots + (x_n + y_n)\vec{a}_n$$

$$= x_1\vec{a}_1 + y_1\vec{a}_1 + \cdots + x_n\vec{a}_n + y_n\vec{a}_n$$

$$= (x_1\vec{a}_1 + \cdots + x_n\vec{a}_n) + (y_1\vec{a}_1 + \cdots + y_n\vec{a}_n)$$

$$= A\vec{x} + A\vec{y} = RHS$$

Proof for (ii)

$$LHS = A\begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix} = (\lambda x_1)\vec{a}_1 + \cdots + (\lambda x_n)\vec{a}_n$$

$$= \lambda(x_1\vec{a}_1) + \cdots + \lambda(x_n\vec{a}_n) = \lambda(x_1\vec{a}_1 + \cdots + x_n\vec{a}_n) = RHS$$

Corollary

Corollary

Let A be a $m \times n$ matrix, $\vec{x}_1, \ldots, \vec{x}_\ell \in \mathbf{R}^\ell$. Then we have

- **(i)** $A(\vec{x}_1 + \cdots + \vec{x}_\ell) = A\vec{x}_1 + \cdots + A\vec{x}_\ell$
- **(ii)** $A(c_1\vec{x}_1 + \cdots + c_\ell\vec{x}_\ell) = c_1(A\vec{x}_1) + \cdots + c_\ell(A\vec{x}_\ell)$

(i) follows from linearity with the aid of induction on ℓ.

Associativity (1)

Thanks to the Linearity, we can prove the following theorem about **Associativity**.

Theorem: Associativity

Given a $m \times n$ matrix A and a $n \times \ell$ matrix X. Then we have for $\vec{c} \in \mathbf{R}^{\ell}$.

$$(AX)\vec{c} = A(X\vec{c})$$

In fact,

$$\begin{aligned} RHS &= A(c_1\vec{x}_1 + \cdots + c_n\vec{x}_n) \\ &= c_1(A\vec{x}_1) + \cdots + c_n(A\vec{x}_n) \\ &= (A\vec{x}_1 \ \ \ldots \ \ A\vec{x}_n)\vec{c} = (AX)\vec{c} = LHS \end{aligned}$$

Associativity (2)

Theorem: Associativity

Given a $m \times n$ matrix A, a $n \times \ell$ matrix B and a $\ell \times t$ matrix C. Then

$$(AB)C = A(BC)$$

(proof)

$$\begin{aligned} LHS &= ((AB)\vec{c}_1 \ \ldots \ (AB)\vec{c}_t) \\ &= (A(B\vec{c}_1) \ \ldots \ A(B\vec{c}_t)) \\ &= A(B\vec{c}_1 \ \ldots \ B\vec{c}_t) \\ &= A(BC) \end{aligned}$$

Multiplication of matrices expressed by rows

In case A is a row vector

In case $A = \mathbf{a} = (a_1 \ \ldots \ a_n)$, we have for a $n \times \ell$ matrix $X = (\vec{x}_1 \ \ldots \ \vec{x}_\ell)$

$$\mathbf{a}X = (\mathbf{a}\vec{x}_1 \ \ldots \ \mathbf{a}\vec{x}_\ell)$$

Multiplication of matrices expressed by rows

On the other hand

$$
\begin{aligned}
& a_1\mathbf{x}_1 + \cdots + a_j\mathbf{x}_j + \cdots + a_n\mathbf{x}_n \\
=& a_1(x_{11} \ \ldots \ x_{1j} \ \ldots \ x_{1\ell}) \\
& \qquad \vdots \\
& + a_i(x_{i1} \ \ldots \ x_{ij} \ \ldots \ x_{i\ell}) \\
& \qquad \vdots \\
& + a_n(x_{n1} \ \ldots \ x_{nj} \ \ldots \ x_{n\ell}) \\
=& (\ldots \ a_1x_{1j} + \cdots + a_ix_{ij} + \cdots + a_nx_{n\ell} \ \ldots) \\
=& (\mathbf{a}\vec{x}_1 \ \ldots \ \mathbf{a}\vec{x}_j \ \ldots \ \mathbf{a}\vec{x}_\ell) \\
=& \mathbf{a}X
\end{aligned}
$$

Multiplication of matrices expressed by rows

In case A is a row vector

For a n dim. row vector $\mathbf{a} = (a_1 \ \dots \ a_n)$ and a $n \times \ell$ matrix

$$X = (\vec{x}_1 \ \dots \ \vec{x}_\ell) = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

we have

$$\begin{aligned} \mathbf{a}X &= (\mathbf{a}\vec{x}_1 \ \dots \ \mathbf{a}\vec{x}_\ell) \\ &= a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n \end{aligned}$$

Multiplication of matrices expressed by rows

Multiplication of matrices

For $m \times n$ matrix $A = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix}$ and $n \times \ell$ matrix we have

$$AX = \begin{pmatrix} \mathbf{a}_1 X \\ \vdots \\ \mathbf{a}_m X \end{pmatrix}$$

In fact

$$AX = \begin{pmatrix} \mathbf{a}_1 \vec{x}_1 & \dots & \mathbf{a}_1 \vec{x}_\ell \\ \vdots & & \vdots \\ \mathbf{a}_i \vec{x}_1 & \dots & \mathbf{a}_i \vec{x}_\ell \\ \vdots & & \vdots \\ \mathbf{a}_m \vec{x}_1 & \dots & \mathbf{a}_m \vec{x}_\ell \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 X \\ \vdots \\ \mathbf{a}_i X \\ \vdots \\ \mathbf{a}_m X \end{pmatrix}$$

Special Matrices

$O_{m,n}$ Zero Matrix

$$O_{m,n} = (\vec{0} \ \ldots \ \vec{0}) = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{pmatrix}$$

is called the zero matrix. It satisfies the identity

$$O_{m,n}X = O_{m,\ell}, \quad YO_{m,n} = O_{k,n}$$

for a $n \times \ell$ matrix X and a $k \times m$ matrix Y.

$O_{m,n}X = O_{m,\ell}$ follows from

$$O_{m,n}\vec{x} = x_1\vec{0} + \cdots + x_n\vec{0} = \vec{0}$$

and $YO_{m,n} = O_{k,n}$ from

$$Y\vec{0} = 0\vec{y}_1 + \cdots + 0\vec{y}_n = \vec{0}$$

Standard Unit Vectors

Standard Unit Vectors

In $\mathbf{R}^3$, we have

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

called the standard unit vectors.

For $m \times 3$ matrix $X = (\vec{x}_1 \ \vec{x}_2 \ \vec{x}_3)$, we have

$$\begin{aligned} X\vec{e}_1 &= 1 \cdot \vec{x}_1 + 0 \cdot \vec{x}_2 + 0 \cdot \vec{x}_3 = \vec{x}_1 \\ X\vec{e}_2 &= 0 \cdot \vec{x}_1 + 1 \cdot \vec{x}_2 + 0 \cdot \vec{x}_3 = \vec{x}_2 \\ X\vec{e}_3 &= 0 \cdot \vec{x}_1 + 0 \cdot \vec{x}_2 + 1 \cdot \vec{x}_3 = \vec{x}_3 \end{aligned}$$

Standard Unit Vectors

Standard Unit Vectors

In $\left(\mathbf{R}^3\right)^*$, we have

$$\mathbf{e}_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

also called the standard unit vectors.

For $3 \times n$ matrix $X = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}$, we have

$$\begin{aligned} \mathbf{e}_1 X &= 1 \cdot \mathbf{x}_1 + 0 \cdot \mathbf{x}_2 + 0 \cdot \mathbf{x}_3 = \mathbf{x}_1 \\ \mathbf{e}_2 X &= 0 \cdot \mathbf{x}_1 + 1 \cdot \mathbf{x}_2 + 0 \cdot \mathbf{x}_3 = \mathbf{x}_2 \\ \mathbf{e}_3 X &= 0 \cdot \mathbf{x}_1 + 0 \cdot \mathbf{x}_2 + 1 \cdot \mathbf{x}_3 = \mathbf{x}_3 \end{aligned}$$

Special Matrices 2

I_n Identity Matrix

$$I_n = (\vec{e}_1 \ \ \dots \ \ \vec{e}_j \ \ \dots \ \ \vec{e}_n) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & 1 \end{pmatrix}$$

is called the **Identity Matrix**. It enjoys, the identity

$$XI_n = X, \quad I_nY = Y$$

for a $m \times n$ matrix X and a $n \times \ell$ matrix Y.

Special Matrices 2

We have the identity

$$(\vec{x}_1 \ \ldots \ \vec{x}_j \ \ldots \ \vec{x}_n)\vec{e}_j = 0 \cdot \vec{x}_1 + \cdots + 1 \cdot \vec{x}_j + \cdots + 0\vec{x}_n = \vec{x}_j$$

This leads us to

$$XI_n = (\vec{x}_1 \ \ldots \ \vec{x}_n)(\vec{e}_1 \ \ldots \vec{e}_n) = (\vec{x}_1 \ \ldots \ \vec{x}_n) = X$$

On the other hand we have, for $\vec{y} \in \mathbf{R}^n$, the identity

$$\vec{y} = y_1\vec{e}_1 + \cdots + y_n\vec{e}_n = I_n\vec{y}$$

Thus we get the identity

$$I_nY = (I_n\vec{y}_1 \ \ldots \ I_n\vec{y}_\ell) = (\vec{y}_1 \ \ldots \ \vec{y}_\ell) = Y$$

Regularity of Matrices, Uniqueness of Inverse

Regularity of Matrices

A $n \times n$ matrix A is called **regular** if there exists another $n \times n$ matrix X satisfying

$$AX = XA = I_n$$

In this situation X is called the **inverse** of A.

- **(i) (Uniqueness of the inverse)** Assume

$$AX = XA = I_n, \quad AY = YA = I_n$$

Then $X = Y$. In fact, from $AX = I_n$ multiplied by Y from the left follows

$$Y(AX) = YI_n = Y$$

On the other hand, $Y(AX) = (YA)X = I_nX = X$.
Accordingly $X = Y$.

Elementary Matrices (1)

To exchange two rows and two columns

$$S_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\vec{e}_2 \ \vec{e}_1 \ \vec{e}_3) = \begin{pmatrix} \mathbf{e}_2 \\ \mathbf{e}_1 \\ \mathbf{e}_3 \end{pmatrix}$$

satisfies

$$S_{12} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \\ \mathbf{c} \end{pmatrix}, \quad (\vec{a} \ \vec{b} \ \vec{c}) S_{12} = (\vec{b} \ \vec{a} \ \vec{c})$$

Elementary Matrices (1)

To exchange two rows and two columns

$$S_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (\vec{e}_3 \ \vec{e}_2 \ \vec{e}_1) = \begin{pmatrix} \mathbf{e}_3 \\ \mathbf{e}_2 \\ \mathbf{e}_1 \end{pmatrix}$$

satisfies

$$S_{13} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{c} \\ \mathbf{b} \\ \mathbf{a} \end{pmatrix}, \quad (\vec{a} \ \vec{b} \ \vec{c}) S_{12} = (\vec{c} \ \vec{b} \ \vec{a})$$

What do we have for

$$S_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = (\vec{e}_1 \ \vec{e}_3 \ \vec{e}_2) = \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_3 \\ \mathbf{e}_2 \end{pmatrix} \quad ?$$

Elementary Matrices (1)

To exchange two rows and two columns

S_{12}, S_{13}, S_{23} are regular.

For example

$$S_{13}S_{13} = S_{13}\begin{pmatrix}\mathbf{e}_3\\ \mathbf{e}_2\\ \mathbf{e}_1\end{pmatrix} = \begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} = I_3$$

Elementary Matrices (2)

To add $\lambda\times$ ith row to jth row

$$R_{21}(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\vec{e}_1 + \lambda\vec{e}_2 \ \vec{e}_2 \ \vec{e}_3) = \begin{pmatrix} \mathbf{e}_1 \\ \lambda\mathbf{e}_1 + \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

satisfies

$$R_{21}(\lambda) \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \lambda\mathbf{a} + \mathbf{b} \\ \mathbf{c} \end{pmatrix}, \quad (\vec{a} \ \vec{b} \ \vec{c})R_{21}(\lambda) = (\vec{a} + \lambda\vec{b} \ \vec{b} \ \vec{c})$$

Elementary Matrices (2)

To add $\lambda\times$ ith row to jth row

$$R_{31}(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix} = (\vec{e}_1 + \lambda\vec{e}_3 \ \vec{e}_2 \ \vec{e}_3) = \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \lambda\mathbf{e}_1 + \mathbf{e}_3 \end{pmatrix}$$

satisfies

$$R_{31}(\lambda)\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \lambda\mathbf{a} + \mathbf{c} \end{pmatrix}, \quad (\vec{a} \ \vec{b} \ \vec{c})R_{31}(\lambda) = (\vec{a} + \lambda\vec{c} \ \vec{b} \ \vec{c})$$

Elementary Matrices (2)

To add $\lambda\times$ ith row to jth row

3×3 matrices $R_{ij}(\lambda)$ $(i \neq j)$ are all regular.

For example we have

$$R_{12}(\lambda)R_{12}(\mu) = R_{12}(\lambda+\mu), \quad R_{12}(0) = I_3$$

Thus we get

$$R_{12}(\lambda)R_{12}(-\lambda) = R_{12}(-\lambda)R_{12}(\lambda) = R_{12}(0) = I_3$$

Elementary Matrices (3)

To multiply jth row by $\lambda \neq 0$

$$Q_2(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\vec{e}_1 \ \lambda\vec{e}_2 \ \vec{e}_3) = \begin{pmatrix} \mathbf{e}_1 \\ \lambda\mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

satisfies

$$Q_2(\lambda) \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \lambda\mathbf{b} \\ \mathbf{c} \end{pmatrix}, \quad (\vec{a} \ \vec{b} \ \vec{c})Q_2(\lambda) = (\vec{a} \ \lambda\vec{b} \ \vec{c})$$

Elementary Matrices (3)

To multiply jth row by $\lambda \neq 0$

$Q_i(\lambda)$ $(i = 1, 2, 3)$ are regular.

In fact we have

$$Q_2(\lambda)Q_2(\mu) = Q_2(\lambda\mu), \quad Q_2(1) = I_3$$

Thus

$$Q_2(\lambda)Q_2(\frac{1}{\lambda}) = Q_2(\frac{1}{\lambda})Q_2(\lambda) = Q_2(1) = I_3$$

Scalar Multiplication to Matirces

Scalar Multiplication to $m \times n$ Matrices

Given a $m \times n$ matix

$$A = (\vec{a}_1 \ \ \ldots \ \ \vec{a}_n) = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix},$$

we define a scalar multiplication by λ to A as follows:

$$\lambda A = (\lambda\vec{a}_1 \ \ \ldots \ \ \lambda\vec{a}_n) = \begin{pmatrix} \lambda\mathbf{a}_1 \\ \vdots \\ \lambda\mathbf{a}_m \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \vdots & & \vdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{pmatrix}$$

Scalar Multiplication to Matirces

Theorem

- **(i)** $(\lambda A)\vec{x} = \lambda(A\vec{x}) = A(\lambda\vec{x})$
- **(ii)** $(\lambda A)X = \lambda(AX) = A(\lambda X)$

The proof for **(i)** is given as follows:

$$\begin{aligned}
(\lambda A)\vec{x} &= (\lambda\vec{a}_1 \ \ \ldots \ \ \lambda\vec{a}_2)\vec{x} \\
&= x_1(\lambda\vec{a}_1) + \cdots + x_n(\lambda\vec{a}_n) = (*) \\
&= \lambda(x_1\vec{a}_1) + \cdots + \lambda(x_n\vec{a}_n) \\
&= \lambda(x_1\vec{a}_1 + \cdots + x_n\vec{a}_n) = \lambda(A\vec{x}) \\
(*) &= (\lambda x_1)\vec{a}_1 + \cdots + (\lambda x_n)\vec{a}_n = A(\lambda\vec{x})
\end{aligned}$$

Moreover the property **(ii)** is derived easily from **(i)**.

Other Basic Properties of Scalar Multiplication

Theorem

- **(iii)** $(\lambda + \mu)A = \lambda A + \mu A$
- **(iv)** $(\lambda\mu)A = \lambda(\mu A)$
- **(v)** $1A = A$ and $0A = O_2$

These properties can derived from the following corresponding properties for vectors. It is necessary to define the addition of matrices to understand (iii), and we put it off for a couple of weeks.

- **(iii)** $(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$
- **(iv)** $(\lambda\mu)\vec{a} = \lambda(\mu\vec{a})$
- **(v)** $1\vec{a} = \vec{a}$ and $0\vec{a} = \vec{0}$