Matrices and their operations No. 2

$m \times n$ Matrices

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$m \times n$ Matrices

$m \times n$ matrices: How to make them

$$A = (\vec{a}_1 \ \ldots \ \vec{a}_n) = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} = \begin{pmatrix} a_{11} & \ldots & a_{1j} & \ldots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \ldots & a_{ij} & \ldots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \ldots & a_{mj} & \ldots & a_{mn} \end{pmatrix}$$

A $m \times n$ matrix is given in the following ways.

- (i) Combining n column vectors $\vec{a}_1, \ \ldots, \ \vec{a}_n \in \mathbf{R}^m$
- (ii) Combining m row vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$
- (iii) Giving $m \times n$ components.

NB a_{ij} is used for the component of the ith row and of the jth column, and sometimes called (i, j) component.

Multiplication: Matrix × Column Vector

Multiplication of n-dim. col. vectors to $m \times n$ matrices

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} = x_1 \vec{a}_1 + \cdots + x_j \vec{a}_j + \cdots + x_n \vec{a}_n = \begin{pmatrix} \mathbf{a}_1 \vec{x} \\ \vdots \\ \mathbf{a}_i \vec{x} \\ \vdots \\ \mathbf{a}_m \vec{x} \end{pmatrix} \in \mathbf{R}^m$$

Here we use multiplication of row vectors and column vectros:

$$\begin{pmatrix} \alpha_1 & \ldots & \alpha_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \alpha_1 x_1 + \cdots + \alpha_n x_n$$

Multiplication: Matrix × Column Vector

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{a}_1 + \cdots + x_j\vec{a}_j + \cdots + x_n\vec{a}_n$$

$$= x_1\begin{pmatrix} a_{11} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + x_j\begin{pmatrix} a_{1j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix} + \cdots + x_n\begin{pmatrix} a_{1n} \\ \vdots \\ a_{in} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} \vdots \\ x_1a_{i1} + \cdots + x_ja_{ij} + \cdots + x_na_{in} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \mathbf{a}_i\vec{x} \\ \vdots \end{pmatrix}$$

Multiplication: Matrix $\times$ Matrix

Multiplication: Matrix $\times$ Matrix

Take another matrix of type $n \times \ell$:

$$X = (\vec{x}_1 \ \ldots \ \vec{x}_\ell) = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

Then a $m \times \ell$ matrix AX is defined by

$$AX = (A\vec{x}_1 \ \ldots \ A\vec{x}_\ell) = \begin{pmatrix} \mathbf{a}_1\vec{x}_1 & \ldots & \mathbf{a}_1\vec{x}_\ell \\ \vdots & & \vdots \\ \mathbf{a}_m\vec{x}_1 & \ldots & \mathbf{a}_m\vec{x}_\ell \end{pmatrix}$$

Linear Map defined by A

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We can define a map

$$\begin{aligned} F_A : \mathbf{R}^n &\longrightarrow \mathbf{R}^m \\ \vec{x} &\mapsto A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n \end{aligned}$$

Linearity of F_A

Linearity of F_A

F_A satisfies the following basic properties called **Linearity**.

- **(i)** $F_A(\vec{x}+\vec{y}) = F_A(\vec{x}) + F_A(\vec{y})$
- **(ii)** $F_A(\lambda\vec{x}) = \lambda F_A(\vec{x})$
- **(iii)** $F_A(\lambda\vec{x}+\mu\vec{y}) = \lambda F_A(\vec{x}) + \mu F_A(\vec{y})$

These three propeties are identical to the following.

- **(i)** $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$
- **(ii)** $A(\lambda\vec{x}) = \lambda(A\vec{x})$
- **(iii)** $A(\lambda\vec{x}+\mu\vec{y}) = \lambda(A\vec{x}) + \mu(A\vec{y})$

Moreover remark that **(iii)** can be easily derived from **(i)** and **(ii)**.

$$A(\lambda\vec{x}+\mu\vec{y}) = A(\lambda\vec{x}) + A(\mu\vec{y}) = \lambda(A\vec{x}) + \mu(A\vec{y})$$

Proof

Proof for (i)

$$LHS = A\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\right) = A\begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

$$= (x_1 + y_1)\vec{a}_1 + \cdots + (x_n + y_n)\vec{a}_n$$

$$= x_1\vec{a}_1 + y_1\vec{a}_1 + \cdots + x_n\vec{a}_n + y_n\vec{a}_n$$

$$= (x_1\vec{a}_1 + \cdots + x_n\vec{a}_n) + (y_1\vec{a}_1 + \cdots + y_n\vec{a}_n)$$

$$= A\vec{x} + A\vec{y} = RHS$$

Proof for (ii)

$$LHS = A\begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix} = (\lambda x_1)\vec{a}_1 + \cdots + (\lambda x_n)\vec{a}_n$$

$$= \lambda(x_1\vec{a}_1) + \cdots + \lambda(x_n\vec{a}_n) = \lambda(x_1\vec{a}_1 + \cdots + x_n\vec{a}_n) = RHS$$

Corollary

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Let A be a $m \times n$ matrix, $\vec{x}_1, \dots, \vec{x}_\ell \in \mathbf{R}^\ell$. Then we have

- **(i)** $A(\vec{x}_1 + \cdots + \vec{x}_\ell) = A\vec{x}_1 + \cdots + A\vec{x}_\ell$
- **(ii)** $A(c_1\vec{x}_1 + \cdots + c_\ell\vec{x}_\ell) = c_1(A\vec{x}_1) + \cdots + c_\ell(A\vec{x}_\ell)$

(i) follows from linearity with the aid of induction on ℓ.

Associativity (1)

Thanks to the Linearity, we can prove the following theorem about **Associativity**.

Theorem: Associativity

Given a $m \times n$ matrix A and a $n \times \ell$ matrix X. Then we have for $\vec{c} \in \mathbf{R}^{\ell}$.

$$(AX)\vec{c} = A(X\vec{c})$$

In fact,

$$\begin{aligned} RHS &= A(c_1\vec{x}_1 + \cdots + c_n\vec{x}_n) \\ &= c_1(A\vec{x}_1) + \cdots + c_n(A\vec{x}_n) \\ &= (A\vec{x}_1 \ \ldots \ A\vec{x}_n)\vec{c} = (AX)\vec{c} = LHS \end{aligned}$$

Associativity (2)

Theorem: Associativity

Given a $m \times n$ matrix A, a $n \times \ell$ matrix B and a $\ell \times t$ matrix C. Then

$$(AB)C = A(BC)$$

(proof)

$$\begin{aligned} LHS &= ((AB)\vec{c}_1 \ \ldots \ (AB)\vec{c}_t) \\ &= (A(B\vec{c}_1) \ \ldots \ A(B\vec{c}_t)) \\ &= A(B\vec{c}_1 \ \ldots \ B\vec{c}_t) \\ &= A(BC) \end{aligned}$$

Multiplication of matrices expressed by rows

In case A is a row vector

In case $A = \mathbf{a} = (a_1 \ \ \ldots \ \ a_n)$, we have for a $n \times \ell$ matrix $X = (\vec{x}_1 \ \ \ldots \ \ \vec{x}_\ell)$

$$\mathbf{a}X = (\mathbf{a}\vec{x}_1 \ \ \ldots \ \ \mathbf{a}\vec{x}_\ell)$$

Multiplication of matrices expressed by rows

On the other hand

$$
\begin{aligned}
& a_1\mathbf{x}_1 + \cdots + a_j\mathbf{x}_j + \cdots + a_n\mathbf{x}_n \\
=& a_1(x_{11} \ \ldots \ x_{1j} \ \ldots \ x_{1\ell}) \\
& \qquad \vdots \\
& + a_i(x_{i1} \ \ldots \ x_{ij} \ \ldots \ x_{i\ell}) \\
& \qquad \vdots \\
& + a_n(x_{n1} \ \ldots \ x_{nj} \ \ldots \ x_{n\ell}) \\
=& (\ldots \ a_1x_{1j} + \cdots + a_ix_{ij} + \cdots + a_nx_{n\ell} \ \ldots) \\
=& (\mathbf{a}\vec{x}_1 \ \ldots \ \mathbf{a}\vec{x}_j \ \ldots \ \mathbf{a}\vec{x}_\ell) \\
=& \mathbf{a}X
\end{aligned}
$$

Multiplication of matrices expressed by rows

In case A is a row vector

For a n dim. row vector $\mathbf{a} = (a_1 \ \ldots \ a_n)$ and a $n \times \ell$ matrix

$$X = (\vec{x}_1 \ \ldots \ \vec{x}_\ell) = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

we have

$$\begin{aligned} \mathbf{a}X &= (\mathbf{a}\vec{x}_1 \ \ldots \ \mathbf{a}\vec{x}_\ell) \\ &= a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n \end{aligned}$$

Multiplication of matrices expressed by rows

Multiplication of matrices

For $m \times n$ matrix $A = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix}$ and $n \times \ell$ matrix we have

$$AX = \begin{pmatrix} \mathbf{a}_1 X \\ \vdots \\ \mathbf{a}_m X \end{pmatrix}$$

In fact

$$AX = \begin{pmatrix} \mathbf{a}_1\vec{x}_1 & \dots & \mathbf{a}_1\vec{x}_\ell \\ \vdots & & \vdots \\ \mathbf{a}_i\vec{x}_1 & \dots & \mathbf{a}_i\vec{x}_\ell \\ \vdots & & \vdots \\ \mathbf{a}_m\vec{x}_1 & \dots & \mathbf{a}_m\vec{x}_\ell \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 X \\ \vdots \\ \mathbf{a}_i X \\ \vdots \\ \mathbf{a}_m X \end{pmatrix}$$