$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$|A| = ad - bc \neq 0$. we have X

$$AX = XA = I_2$$

$$X = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1}$$

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad |A| = \cos^2\theta + \sin^2\theta = 1 \neq 0.$$

$$A^{-1} = \frac{1}{1} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Rotation Matrix.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta+\alpha) & -\sin(\theta+\alpha) \\ \sin(\theta+\alpha) & \cos(\theta+\alpha) \end{pmatrix}$$

$\uparrow$

important

Use Addition Theorem for cos and sin.

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

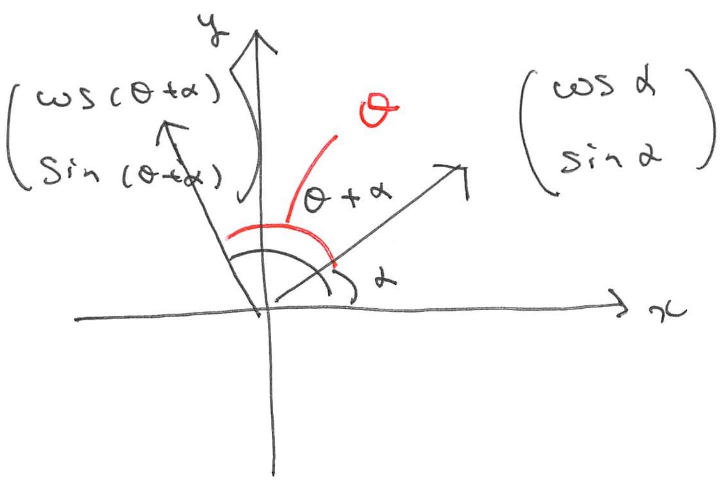
$$R_\theta \cdot R_\alpha = R_{\alpha+\theta}.$$

$$R_0 = \begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

$$R_\theta \cdot R_{-\theta} = R_{-\theta} \cdot R_\theta = R_0 = I_2.$$

$$(R_\theta)^{-1} = R_{-\theta}. = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} \cos(\theta+\alpha) \\ \sin(\theta+\alpha) \end{pmatrix}$$



Example

Let $A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$. Then by C-H we have

$|A| = 5 \cdot 2 - 2 \cdot 2 = 6.$

$$A^2 - 7A + 6I_2 = O_2$$

We associate the eigen-polynomial to A by

$$\Phi_A(\lambda) = \lambda^2 - 7\lambda + 6 = (\lambda - 1)(\lambda - 6).$$

The factorization above tells us that $1 + 6 = 7$ and $1 \cdot 6 = 6$. Thus the equation deduced from C-H is written by

$$A^2 - (1+6)A + 1 \cdot 6 \, I_2 = O_2$$

which can be interpreted in two ways by

$$A(A - I_2) = 6(A - I_2), \quad A(A - 6I_2) = (A - 6I_2),$$

$A^2 - A = 6A - 6I_2 = 6(A - I_2)$

$\shortparallel$

$A(A - I_2)$

$A^2 - 6A = A - 6I_2 \longrightarrow A(A - 6I_2) = A - 6I_2.$

Example(2)

We try to find $A^n(A - I_2)$ and $A^n(A - 6I_2)$.

$$\begin{aligned} A^2(A - I_2) &= A \cdot A(A - I_2) = A \cdot 6(A - I_2) = 6 \cdot A(A - I_2) \\ &= 6 \cdot 6(A - I_2) = 6^2(A - I_2) \\ A^3(A - I_2) &= A \cdot A^2(A - I_2) = A \cdot 6^2(A - I_2) = 6^2 \cdot A(A - I_2) \\ &= 6^2 \cdot 6(A - I_2) = 6^3 (A - I_2) \\ &\vdots \\ A^n(A - I_2) &= 6^n(A - I_2) \end{aligned}$$

We find that

$$A^n(A - I_2) = 6^n(A - 6I_2)$$

We can find more easily

$$A^n(A - 6I_2) = (A - 6I_2)$$

Example (3)

In the previous slides we have shown

$$A^n(A - I_2) = 6^n(A - I_2), \quad A^n(A - 6I_2) = (A - I_2)$$

6

which is views as

$$\begin{aligned} A^{n+1} - A^n &= 6^n(A - I_2) \quad (1) \\ A^{n+1} - 6A^n &= (A - 6I_2) \quad (2) \end{aligned}$$

Finally (1)–(2) is

$$5A^n = 6^n(A - I_2) - (A - 6I_2) \quad i.e. \quad A^n = \frac{6^n}{5}(A - I_2) - \frac{1}{5}(A - 6I_2)$$

Theorem

$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Theorem

Let $A \in M_2(\mathbf{R})$ be a 2×2 matrix and assume its eigen polynomial has two simple roots:

$$\begin{aligned} \Phi_A(\lambda) &= \lambda^2 - (a+d)\lambda + (ad - bc) \cancel{= (\lambda - \alpha)} \\ &= (\lambda - \alpha)(\lambda - beta) \quad \text{with} \quad \alpha \neq \beta \end{aligned}$$

β

Then A^n is in the form

$$A^n = \alpha^n X_0 + \beta^n X_1$$

with $X_0, X_1 \in M_2(\mathbf{R})$ independent of n.

Polynomials of matrices

We can form polynomials in the matrix $A \in M_2(\mathbf{R})$.

Definition

For any polynomial

$$f(\lambda) = a_n\lambda^n + \cdots + a_1\lambda + a_0 \qquad a_j \in \mathbb{R}$$

we define

$$f(A) := a_0 A^n + \cdots + a_1 A + a_0 I_2$$

Basic properties of this operation is given in the following theorem.

Theorem

For f, g polynomials in λ, we have

$$(f+g)(A) = f(A) + g(A), \quad (f \cdot g)(A) = f(A) \cdot g(A)$$

Multiplication in $M_2(\mathbb{R})$

multiplication as polynomial.

$$f(\lambda) = \lambda^2 - \lambda + 2$$

$$f(A) = A^2 - A + 2I_2$$

Eigenpolynomials

Definition and a remark

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, wedefine its *eigenpolynomial* by $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\Phi_A(\lambda) = |\lambda I_2 - A| = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - b \end{vmatrix} \quad d.$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

$$= \lambda^2 - (a+d)\lambda + |A|$$

$$= (\lambda - a)(\lambda - d) - bc.$$

By Theorem of C-H we have

$$\Phi_A(A) = O_2$$

$(A-2I_2)(A-2I_2) = (A^2-4A+4I_2 = O_2$ by C-H.

Application

$|A| = 1\cdot 3 - 1\cdot(-1) = 4$

Let $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$. Then we have by Theorem of C-H

$\Phi_A(\lambda) = \lambda^2 - 4\lambda + 4$
$= (\lambda-2)^2$

$$A^2 - 4A + 4I_2 = O_2 \quad i.e. \quad (A-2I_2)^2 = O_2$$

We divide λ^n by $(\lambda-2)^2$. Then there exist a polynomial $q(\lambda)$ and $a, b \in \mathbf{R}$ satisfying

$$\lambda^n = q(\lambda)(\lambda-2)^2 + a\lambda + b \tag{1}$$

We get the derivative of this equation to get

$$n\lambda^{n-1} = q'(\lambda)(\lambda-2)^2 + q(\lambda)\cancel{(\lambda-2)^2} + a \tag{2}$$

$2(\lambda-2)$

Application

We substitute 2 for λ in (1) to get

$b = 2^n - 2\cdot n 2^{n-1}$

$$2^n = 2a + b \tag{3}$$

and in (2) to get

$$n2^{n-1} = a \tag{4}$$

Thus we get

$$a = n2^{n-1}, \quad b = (1-n)2^n \tag{5}$$

Accordingly we have

$$\lambda^n = q(\lambda)(\lambda-2)^2 + n2^{n-1}\lambda + (1-n)2^n$$

We substitute A for λ to derive

$$A^n = q(A)\underbrace{(A-2I_2)^2}_{= O_2} + n2^{n-1}A + (1-n)2^n I_2 = n2^{n-1}A + (1-n)2^n I_2$$

Matrices and their operations No. 2

$m \times n$ Matrices

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$m \times n$ Matrices

$m \times n$ matrices: How to make them

$$A = (\vec{a}_1 \ \dots \ \vec{a}_n) = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix} \begin{array}{l} \\ \\ \text{ith row} \\ \\ \\ \end{array}$$

jth column.

A $m \times n$ matrix is given in the following ways.

- (i) Combining n column vectors $\vec{a}_1, \dots, \vec{a}_n \in \mathbf{R}^m$
- (ii) Combining m row vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$
- (iii) Giving $m \times n$ components.

NB a_{ij} is used for the component of the ith row and of the jth column, and sometimes called (i, j) component.

$$A = (\vec{a}_1 \cdots \vec{a}_n) = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix}$$

Multiplication: Matrix × Column Vector

Multiplication of n-dim. col. vectors to $m \times n$ matrices

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{a}_1 + \cdots + x_j\vec{a}_j + \cdots + x_n\vec{a}_n = \begin{pmatrix} \mathbf{a}_1\vec{x} \\ \vdots \\ \mathbf{a}_i\vec{x} \\ \vdots \\ \mathbf{a}_m\vec{x} \end{pmatrix} \in \mathbf{R}^m$$

Here we use multiplication of row vectors and column vectros:

$$(\alpha_1 \ \ldots \ \alpha_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \alpha_1 x_1 + \cdots + \alpha_n x_n$$

$$(\vec{a}\ \vec{b}\ \vec{c}) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\vec{a} + y\vec{b} + z\vec{c}$$

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + y\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + z\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Multiplication: Matrix × Column Vector

$$= \begin{pmatrix} x a_1 + y b_1 + z c_1 \\ x a_2 + y b_2 + z c_2 \\ x a_3 + y b_3 + z c_3 \end{pmatrix}$$

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{a}_1 + \cdots + x_j\vec{a}_j + \cdots + x_n\vec{a}_n$$

ith component

$$= x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + x_j \begin{pmatrix} a_{1j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{in} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} \vdots \\ x_1 a_{i1} + \cdots + x_j a_{ij} + \cdots + x_n a_{in} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \mathbf{a}_i\vec{x} \\ \vdots \end{pmatrix}$$

$$(a_2\ b_2\ c_2) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$(a_{i1}\ a_{i2}\ \cdots\ a_{ij}\ \cdots\ a_{in}) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Multiplication: Matrix × Matrix

Multiplication: Matrix × Matrix

Take another matrix of type $n \times \ell$:

$$X = (\vec{x}_1 \ \ldots \ \underbrace{\vec{x}_\ell}_{=\ \ell}) = \underbrace{\begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}}_{=\ n}$$

Then a $m \times \ell$ matrix AX is defined by

$$AX = (A\vec{x}_1 \ \ldots \ A\vec{x}_\ell) = \begin{pmatrix} \mathbf{a}_1\vec{x}_1 & \ldots & \mathbf{a}_1\vec{x}_\ell \\ \vdots & & \vdots \\ \mathbf{a}_m\vec{x}_1 & \ldots & \mathbf{a}_m\vec{x}_\ell \end{pmatrix}$$

m rows ℓ colums

$A : m \times n$ m rows, n colums

$X : n \times \ell$ n rows, ℓ colums

$\rightarrow$ AX m rows, ℓ colums.

Linear Map defined by A

Linear Map defined by A

We can define a map

$$\begin{aligned} F_A : \mathbf{R}^n &\longrightarrow \mathbf{R}^m \\ \vec{x} &\mapsto A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n \end{aligned}$$

$$\begin{pmatrix} \boxed{1 \quad 0 \quad \lambda} \\ \boxed{0 \quad 1 \quad 0} \\ \boxed{0 \quad 0 \quad 1} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + \lambda z \\ y \\ z \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}} \begin{pmatrix} x & p \\ y & q \\ z & r \end{pmatrix} = \begin{pmatrix} z & r \\ y & q \\ x & p \end{pmatrix}$$

Elementary Matrix.